Introduction to Dessins d'Enfants and the Grothendieck Correspondence.

Mémoire présenté en vue de l'obtention du titre de licencié en Sciences Mathématiques.

Jimmy Dillies
Promoteur:
Juan Carlos Alvarez Paiva

Université Catholique de Louvain
Département de Mathématiques
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Remerciements

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Merci enfin à papa, maman et Emeline car pour faire un arbre ... Mon Dieu que c'est long !

Foreword

Why a mémoire on Dessins d'Enfants?

Dessins d'enfants form a relatively new area of research in Mathematics but are far from being totally mastered. Their richness lies in their various interconnections with other areas such as topology, algebraic geometry, groups,complex analysis or even mathematical physics. A lot of articles have already been published around the subject but, alas, just three books ([**15**],[**21**] and [**16**]) - that are nothing more than collections of articles - are available.

The aim I set myself was to make a short introduction open to all people that would be interested in the subject. I tried to make it as much self contained as possible so that no important requirements in the pre-cited topics are necessary.

Hoping you'll discover a nice subject and enjoy it as I did, have a good read.

Jimmy
Warwick, May 2000

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Part 1

The different faces of dessins d'enfants.

CHAPTER 1

Definition of Dessins d'Enfants

1.1. Heuristic presentation

Dessins d'enfants have been introduced by Grothendieck in order to give a new point of view to the study of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and generally algebraic geometry. The aim of this new point of view was a new attack of the subject without any advanced background necessary. As he writes it in his esquisse d'un programme in 1984 (part of his application to the CNRS) the simplicity of these objects are really far from the deepness of the results that could be hoped to be found.

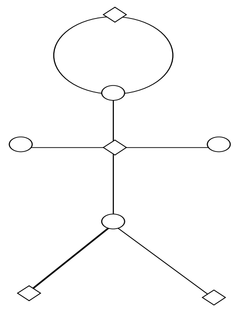
> *"Je ne crois pas qu'un fait mathématique m'ait jamais autant frappé que celui-là, et ait eu un impact psychologique comparable. Cela tient sûrement à la nature tellement familière, non technique, des objets considérés, dont tout* dessin d'enfant *griffoné sur un bout de papier (...) donne un exemple parfaitement explicite"*
>
> Alexandre Grothendieck, Esquisse d'un Programme 1984[1]

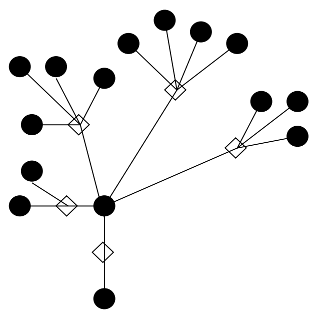
Indeed, in a plane dessins d'enfants or children drawings can be seen as the result of the next construction.

1. Draw in the plane points of two types
2. Join the points with the condition that two points of the same type can't be connected and lines can't cross each other.

This can give us results like the following that are nothing else than what a kid could just paint[2].

[1] "I don't believe that any other mathematical fact impressed me so much and had such a psychological influence. It certainly holds in the so familiar nature, un-technical, of the considered objects whose image could be given by a child's drawing on a piece of paper."
[2] compare with cover ...





NOTE. In the literature these occurences will often be denoted as "*petit bonhomme*" and "*Leila's flowers*". It is this simplistic idea of this objects that 'shocked' Grothendieck so deeply.

Based on a important theorem of Belyi, Grothendieck will associate the combinatorial properties of our drawings to much more advanced items that are coverings.

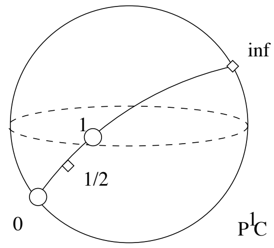
So, let us dive into the deep world of dessins d'enfants.

1.2. Definitions.

However it is nice to work in the plane, the natural environment of dessins will be the world of compact surfaces and dessins will be 'graph-theoretically' defined.

DEFINITION 1.2.1. A *Grothendieck dessin* is a triple $X_0 \subset X_1 \subset X_2$ where X_2 is the topological model of a connected compact Riemann surface, X_0 is a finite set of points and $X_1 \setminus X_0$ is a finite disjoint union of open segments, $X_2 \setminus X_1$ is a finite disjoint union of open cells, such that a bipartite structure can be put on the set of vertices. That is, vertices can be coloured with two different marks in such a way that direct neighbours are from the opposite marks.

EXAMPLE 1.2.2. $(\{0, \frac{1}{2}, 1, \infty\}, [0, \infty], \mathbb{P}^1\mathbb{C})$ is an alternated linear tree of length 3 on the Riemann sphere.



It is true this first definition already gives a more precise idea of a dessin but the interesting parts of a dessin are its combinatorial properties, not the way it is embedded. We must so define a class of equivalence between dessins.

DEFINITION 1.2.3. An *abstract dessin* is an isomorphism class of Grothendieck dessins under the relation : two dessins (X_0, X_1, X_2); $(\tilde{X}_0, \tilde{X}_1, \tilde{X}_2)$ are isomorphic if there's a homeomorphism $X_2 \to \tilde{X}_2$ that induces a homeophormism $X_1 \to \tilde{X}_1$ and $X_0 \to \tilde{X}_0$.

EXAMPLE 1.2.4. The dessin in the previous example is in the same isomorphism class as the dessin $(\{0, 1, 2, \infty\}, [0, \infty], \mathbb{P}^1\mathbb{C})$. The homeomorphism used here is the homotethy $\phi\, \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C} : x \mapsto 2x$.

Another classification of dessin we will also need later is the notion of pre-clean dessin[3].

DEFINITION 1.2.5. A *pre-clean Grothendieck dessin* is a triple $X_0 \subset X_1 \subset X_2$ where X_2 is the topological model of a connected compact Riemann surface, X_0 is a finite set of points and $X_1 \setminus X_0$ is a finite disjoint union of open segments, $X_2 \setminus X_1$ is a finite disjoint union of open cells. [4]

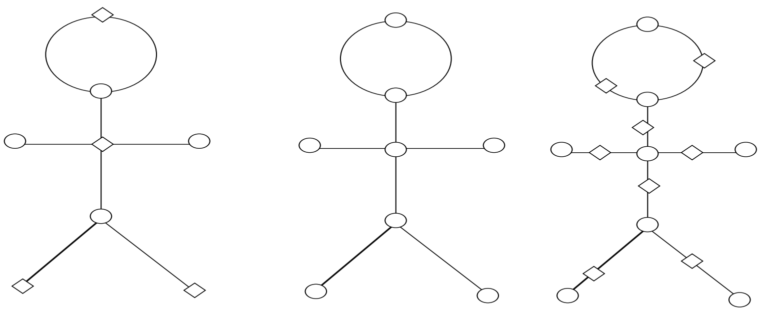
REMARK. This definition of pre-clean Grothendieck dessin seems weaker than Grothendieck dessin. In fact, in is the opposite. With a pre-clean dessin we must (unconsciously) associate to each of its edges a point of a second type of marking. So every point of the second mark has degree two. It is thus

[3]People as Schneps or Zaponi use the term *pre-clean* while others like Granboulan use the term *clean*.
[4]Some will have recognized the definition of cellular graph.

clear that this definition is more restrictive. We will thus also call *pre-clean dessin* a dessin where the elements of one mark have degree two or less.[5]

One can remark that to each dessin can be associated a pre-clean dessin by 'freezing' (=identifying) its marks and then to each edge associate a second-type vertex. We could of course also 'triple' or 'quadruple' a dessin.

EXAMPLE 1.2.6. Dubbling of the *Petit Bonhomme*.



The evolutions are :

- original dessin
- freezing of marks
- new dessin

We have now introduced the main definition of dessin d'enfant. It is the simplest and it permits us to draw dessins in a very intuitive way. But this defition is equivalent to several others. Many of those were already known a 'long' time ago. Before going further it will be interesting to take a look at the other faces of dessins d'enfants...

[5]If we prefer to use the terminology pre-clean it is because we can so hold the term clean for dessins where vertices of type 2 have exactly degree 2.

CHAPTER 2

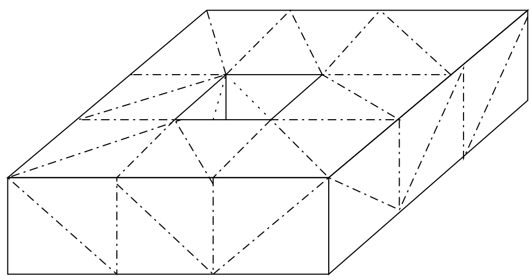
The other faces of dessins d'enfants.

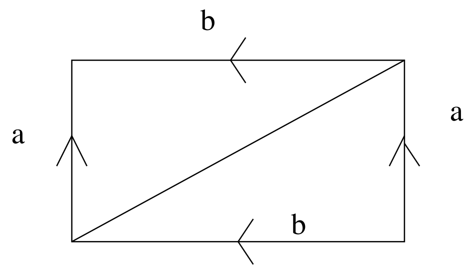
From now on we will implicitly work on oriented surfaces - still compact.

2.1. Triangulations.

DEFINITION 2.1.1. A *triangulation* is a cellular graph X_0, X_1, X_2 where each open cell is planar bounded by exactly three edges.

EXAMPLE 2.1.2. A triangulation of the torus and the smallest triangulation of a torus.





Alternative 1

DEFINITION 2.1.3. A *bicolored triangulation* is a triangulation X_0, X_1, X_2 with a map $S : X_2 \setminus X_1 \to \{0, 1\}$ such that neighbour cells have different images.[1]

Before showing that bicolored maps correspond to dessin we will introduce a second alternative and show the equivalence between these. Alternative 2

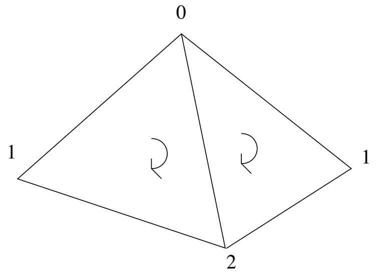
[1]Note that only oriented surfaces can have a bicoloured triangulation.

DEFINITION 2.1.4. A *tripartite triangulation* is triangulation X_0, X_1, X_2 with a map $T : X_0 \to \{0, 1, 2\}$ such that the edges are bordered by vertices of different image.[2]

PROPOSITION 2.1.5. *Tripartite and bicolored triangulations are in bijection.*

PROOF. Let $X_0, X_1, X_2; T$ be a tripartite map. X_2 is orientated we can so define a clock-wise direction. For each cell we read in that order the images of its border vertices and put it in a triple P. We define the bipartite map S by sending a cell to 0 if the parity of the permutation sending P to $(0, 1, 2)$ is even and to 1 otherwise. Note that starting from a different point doesn't affect the relative parity of a cell.

We now have to show that neighbour cells have different images through S. It is clear that renaming the elements of X_0 doesn't change the relative parity of neighbour cells. We can thus consider the next picture, showing neighbour cells, without loss of generality.



We see clearly that these have opposite parity.

Let's take now $X_0, X_1, X_2; S$ a bicolored map. To get a map T we first fix a cell B. We assign to each of its summits a different value out of 0,1,2. Then, by induction we fix the value of the other summits. The bicoloration of our map implies that each element of X_0 is common to an even number of cells and thus our construction makes sense. □

We must now see that these presentations are equivalent to dessins.

PROPOSITION 2.1.6. *Alternative 1 and 2 are equivalent to pre-clean dessins.*

PROOF. By last proposition it is sufficent to prove bijection between tripartite triangulations and pre-clean dessins.

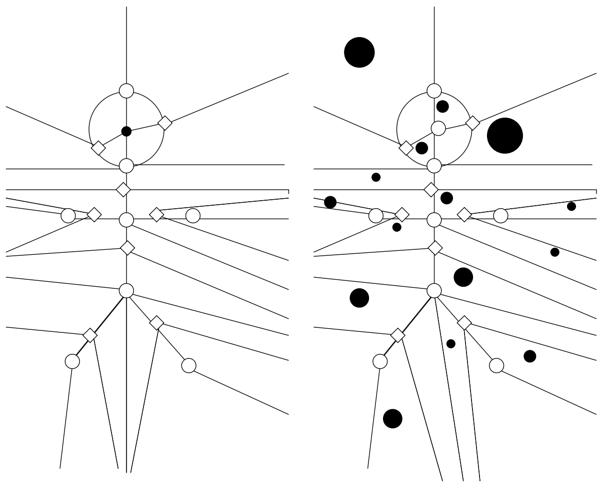
[2]On the graphics we'll use the bijectivity of $\{0, 1, 2\}$ and any tree symbol set to visualize T.

First let us take a pre-clean dessin where the points of both marks are drawn. In each cell we take a point that we denote by a third mark. Now joining all these third type points to those of the first two classes laying on its edges we get a tripartite triangulation.

The other way round we chose two marks and we take the restriction of the tripartite triangulation to the edges not touching third mark points. The third marks remaining lie in the middle of the cell made of the triangles of which it was a summit. □

REMARK. The second part of the proof doesn't fix one dessin but possibly three. To fix a one-to-one correspondance we must choose to always take the same first two marks to become the basis of the skeleton of the dessin.

EXAMPLE 2.1.7. We have here our pre-clean petit bonhomme seen as a tripartite map and a bicolored map. We can remark that on the tripartite map the point in the outside cell is sent to infinity (we are on $\mathbb{P}^1\mathbb{C}$).



DEFINITION 2.1.8. The *degree of a dessin* is the number of it's edges. Seen as triangulation it is half of the number of triangles.

We will usually denote it here as N.

DEFINITION 2.1.9. The *genus of a dessin* is the genus on the surface on which it is embedded.

We will denote it by g. Two similar dessins (dessins homeomorphic when considered on a open subset of the surface) can thus have different genuses because of their 'X_2'. We have now an easy lemma to calculate the genus of a dessin.

LEMMA 2.1.10. *The genus of a dessin is*

$$g = \frac{1}{2}[2 + N - \sum_{i=0}^{2} x_i]$$

where the x_i are the number of first, second and third type mark points considered on the relevant tripartite map.

PROOF. The tripartite triangulation is a triangulation of X_2. It's genus (what we are calculating, by definition) is

$$g = \frac{2-\chi}{2}$$

where χ is it's euler characteristic. We have that

$$\chi = c - e + v$$

where c,e and v are respectively the number of cells, edges and vertices of the triangulation. We have $2N = e$ and $\sum x_i = v$. So $\chi = 2N - e + \sum x_i$. So we have to show that

$$e = 3N \tag{2.1.1}$$

That is clear : each cell is determined by 3 edges, so for each cell we take one edge but that edge also determines the symetric triangle... so $e.2 = 3.c$ that is 2.1.1. □

EXAMPLE 2.1.11. Let's calculate the genus of our petit bonhomme on $\mathbb{P}^1\mathbb{C}$. We have

$$\begin{cases} N & = & 15 \\ \sum_{j=0}^{2} x_j & = & 17 \end{cases}$$

And so the genus of the dessin is 0.

Before going to the group addicted view of dessins we'll need two other different and very basic points of view of dessins. We use the pre-clean dessins as defined above.

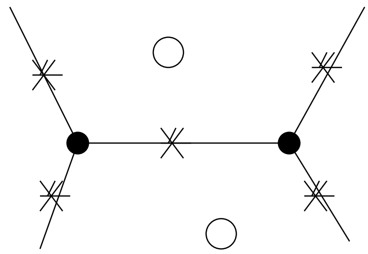
DEFINITION 2.1.12. A *marking* on a pre-clean dessin is a fixed choice of one point on each component of $X_1 \setminus X_0$, and one point in each open cell of

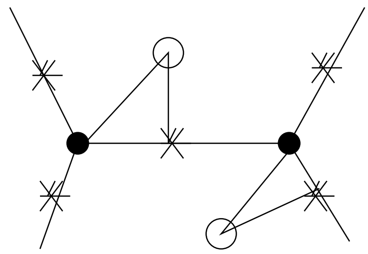
$X_2 \setminus X_1$. We will use the notation $\bullet$ for a point of X_0,$\star$ for a point in $X_1 \setminus X_0$ and $\circ$ for a point in $X_2 \setminus X_1$.

.

DEFINITION 2.1.13. Let D be a pre-clean dessin with fixed marking. Then the *flag set* F(D) of D is the set of triangles whose three vertices are marked $\star, \bullet \circ$ with $\bullet$ in the closure of the segment on which $\star$ lies and with this same segment in the closure of the cell in which $\circ$ lies. The *oriented flag set* $F^+(D)$ is the set of flags where the order of the vertices is $\star, \bullet \circ$ when read clockwise.

EXAMPLE 2.1.14. Here's a part of marked dessin and the same dessin with two elements of F(D).





REMARK. It is of course possible when regarding the whole dessins that the two $\circ$ are associated.

We have now seen the main definitions concerning dessins, that have a 'topological' taste. We can now switch to a more algebraic part with the study of groups.

2.2. Groups

The groups we'll use will represent the dessin or will act on it. By act we mean

Definition 2.2.1. The *action* of a group G on a set E is a homomorphism between G and $Aut(E)$.

The groups that will represent our dessins are in fact not so far from topology as their names indicate: cartographical group, ...

Definition 2.2.2. The *cartographical group* $\mathfrak{C}_2$ is given by

$$\mathfrak{C}_2 =< \sigma_0, \sigma_1, \sigma_2 \left| \sigma_i^2 = (\sigma_0\sigma_2)^2 = 1 \right. >$$

It has a subgroup of order 2 called *the oriented cartographical group* given by all even words of $\mathfrak{C}$. We denote it $\mathfrak{C}_2^+$. Generatively it can be rewritten as

$$\mathfrak{C}_2^+ =< \rho_0, \rho_1, \rho_2 \left| \rho_1^2 = \rho_0\rho_1\rho_2 = 1 \right. >$$

where

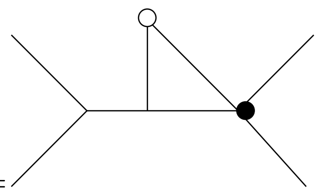
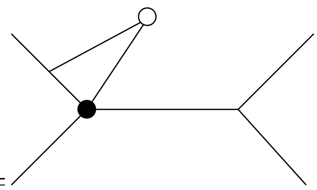
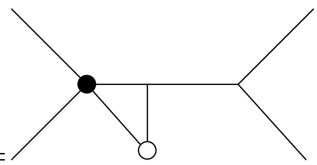
$$\rho_0 = \sigma_1\sigma_0 \; \rho_1 = \sigma_0\sigma_2 \; \rho_2 = \sigma_2\sigma_1$$

We will see that the conjugacy classes of $\mathfrak{C}_2^+$ are in some way in bijection with clean dessins. Therefore we will first describe the action of $\mathfrak{C}_2$ on the set of flags of a clean dessin D.

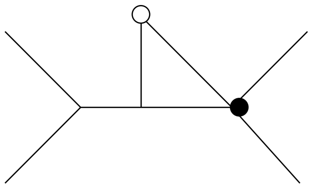
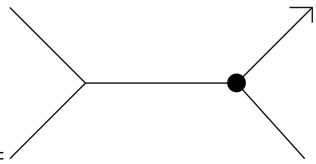
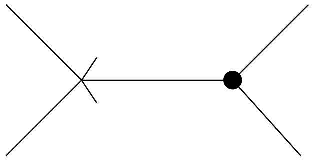
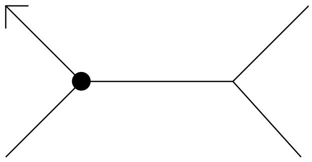
The σ_i generators of the cartographical group act on F(D) as follows:

Let's take a flag F

Then we have

- $\sigma_0(F) =$ 
- $\sigma_1(F) =$ 
- $\sigma_2(F) =$ 

It is easy to see that the action is faithful. We can restrict our attention to oriented flags who can be denoted by just the $\bullet - \star$ edge. In fact it is even sufficent to denote for each flag a $\bullet$ and an arrow in the direction of the $\star$. For the flag shown as above (which is 'conveniently' oriented) we can also observe its image under the action of the ρ_i. We get here:

- $\rho_0(F) = (\sigma_1\sigma_0)(F) = \sigma_1($  $)=$ 
- $\rho_1(F) =$ 
- $\rho_2(F) =$ 

REMARK. In the decomposition of ρ_i we can not use the arrow representation since the σ_i send elements from $F^+ \to F \setminus F^+$ and vice-versa.

The study of the dessin through $F^+(D)$ under the action of $\mathfrak{C}_2^+$ makes the marking unnecessary since each oriented flag is in one-to-one correspondance with pairs composed of an edge - one of its boundary vertex.

LEMMA 2.2.3. *Let D be a pre-clean dessin and $F \in F^+(D)$ a fixed flag. Let $B_{F,D}$ be the set of elements of $\mathfrak{C}_2^+$ fixing F. Then $B_{F,D}$ is a subgroup of finite index in $\mathfrak{C}_2^+$ and the stabilizing subgroup $B_{F',D}$ for any other flag $F' \in F^+(D)$ is conjugate to $B_{F,D}$ in $\mathfrak{C}_2^+$. Moreover $B_{F,D}$ depends only on the abstract dessin D.*

PROOF. It is clear that $B_{F,D}$ is a subgroup of $\mathfrak{C}_2^+$. Since F(D) is finite so is the orbit of F and thus $B_{F,D}$ is of finite index.

Let's take F' in $F^+(D)$ different from F. By applying the different transformations of $\mathfrak{C}_2^+$ to F we can send it onto F'. So we have an element $\mu \in \mathfrak{C}_2^+$ such that $\mu(F) = F'$ and so $B_{F,D} = \mu^{-1}B_{F',D}\mu$. □

THEOREM 2.2.4. *(Malgoire-Voisin) There's a bijection between the isomorphism classes of clean dessins and the conjugacy classes of subgroups of $\mathfrak{C}_2^+$ of finite index.* Alternative 3

PROOF. By the lemma therefore we know that to each abstract dessin there is a corresponding conjugacy class of subgroups of finite index of $\mathfrak{C}_2^+$. Now, the other way round, we have, given B finite subgroup of $\mathfrak{C}_2^+$, to construct a unique dessin (abstract) from it.

We consider B as a finite index subgroup of $\mathfrak{C}_2$ and we denote by H the coset space $H = \mathfrak{C}_2/B$. We will construct a dessin whose flag set F(D) will be bijective to H. So $F^+(D)$ will be bijective to $\mathfrak{C}_2^+/B$. We will also take care that the action of $\mathfrak{C}_2$ on F(D) will be the same as it's action on H. The flag F corresponding to B will be defined by the action of B. We start by looking at the action of σ_1 and σ_2 on $< \sigma_0 > \backslash H$. Since two elements of our expected dessin are in the same σ_0 orbit if they have the $\circ - \star$ edge in common, we can identify the $< \sigma_0 > \backslash H$ elements with the $\circ - \star$ edges. So for elements of $< \sigma_0 > \backslash H$ to be in the same σ_1 (respectively σ_2) orbit they must have the $\circ$ (resp. $\bullet$) in common. We can apply the same argument to $< \sigma_1 > \backslash H$ and $< \sigma_2 > \backslash H$, this gives us this summary table □

Group	Elements	σ_0 orbit denominator	σ_1 ...	σ_2 ...
$< \sigma_0 > \backslash H$	$\circ - \star$	X	$\circ$	$\star$
$< \sigma_1 > \backslash H$	$\circ - \bullet$	$\circ$	X	$\bullet$
$< \sigma_2 > \backslash H$	$\bullet - \star$	$\star$	$\bullet$	X

We are now going to give the points, edges and open cell the order of the double orbit sets$< \sigma_1, \sigma_2 > \backslash H$, $< \sigma_0, \sigma_2 > \backslash H$ and $< \sigma_1, \sigma_0 > \backslash H$ respectively . We now have to glue everything together. We take a point and an edge by taking an element in $< \sigma_1, \sigma_2 > \backslash H$ and $< \sigma_0, \sigma_2 > \backslash H$ respectively. Elements of these two elements can be seen as composed σ_1, σ_2 and σ_0, σ_2 orbits. Elements of our two sets can be 'glued' together if they have a common σ_2 orbit. We do the same for all cross products available with our three sets to glue all elements together. The situation can be synthesised in these two graphs where the $F_{\star - \bullet}$ denote the set of $\star - \bullet$ and so on. The $F_\star$ denote the set of edges and so on. The parallel between the next diagrams describes the situation very well.

$$
\begin{array}{ccccc}
 & & F(D) & & \\
 & \swarrow & \downarrow & \searrow & \\
F_{\circ-\star} & & F_{\circ-\bullet} & & F_{\star-\bullet} \\
\downarrow & \searrow\swarrow & & \searrow\swarrow & \downarrow \\
F_{\circ} & & F_{\star} & & F_{\bullet} \\
 & \searrow & \downarrow & \swarrow & \\
 & & \cdot & &
\end{array}
$$

$$
\begin{array}{ccccc}
 & & H & & \\
 & \swarrow & \downarrow & \searrow & \\
<\sigma_0>\backslash H & & <\sigma_1>\backslash H & & <\sigma_2>\backslash H \\
\downarrow & \searrow\swarrow & & \searrow\swarrow & \downarrow \\
<\sigma_0,\sigma_1>\backslash H & & <\sigma_0,\sigma_2>\backslash H & & <\sigma_1,\sigma_2>\backslash H \\
 & \searrow & \downarrow & \swarrow & \\
 & & 1 & &
\end{array}
$$

We have to note that the dessin is independent from the choice of representative of the conjugacy class of B. Indeed the action of $\mathfrak{C}_2$ on H/B and $H/\sigma^{-1}B\sigma$ is the same for $\sigma \in \mathfrak{C}_2^+$ and so changing B into $\sigma^{-1}B\sigma$ doesn't change any of the objects above. Considering subsets of $\mathfrak{C}_2^+$ acting on H and remembering the last lemma we can conclude. So to get isomorphic dessins we must take a conjugate subgroup to B in $\mathfrak{C}_2$.

Remark. The easiest way to generate the group given the triangulation, is to remember that σ_0 decomposes into cycles that are a representation of the vertices together with their multuplicity; that σ_1 decomposes into A[3] cycles of length (permutation of the cells around each vertex)[4] two and that $\sigma_0\sigma_1\sigma_2 = 1$.

Example 2.2.5. The torus (see picture on first page of chapter two):

1. $\sigma_0 = (1,4,2,5,3,6)$
2. $\sigma_1 = (1,5)(3,4)(2,6)$
3. $\sigma_\infty = (\sigma_0\sigma_1)^{-1}$

Where 1,2,3,.... are the edges of the triangles. ($\triangle$ we consider each edge twice !)

[3]number of edges
[4]for Bauer and Itzykson this is swapping the paired arrows of thick graphs (cf. **[2]**)

We have now seen the main three alternative ways of considering a dessin. These points of view are often the first face encountered by mathematicians not studying dessins for their own sake.

We now switch over to The Fourth alternative, the one that made dessins interesting. This part will be discussed in Part II.

CHAPTER 3

And the Grothendieck Correspondance

3.1. Small reminder.

Some definitions written here will be used in later chapters.

- We say that an algebraic variety V is *defined* on $\mathbb{K}$, a field, if $I(V)$ is generated by $I(V) \cap \mathbb{K}[X]$.
- A *number field* $\mathbb{K}$ is a finite extension of $\mathbb{Q}$.
- All elements of $\mathbb{C}$ that generate a finite extension $\mathbb{Q}(a)$ are called algebraic numbers.
- Each number field $\mathbb{K}$ is isomorphic to $\mathbb{Q}[X]/(P)$ where $deg(P) = [\mathbb{K} : \mathbb{Q}]$. P has at least one solution in $\mathbb{K}$, defined as a primitive element of $\mathbb{K}$.
- $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is the field whose elements are all the algebraic numbers.

So we can write

$$\overline{\mathbb{Q}} = \bigcup_{\mathbb{K} \in \mathfrak{K}} \mathbb{K}$$

where $\mathfrak{K}$ is the set of all galois extensions of $\mathbb{Q}$ in $\mathbb{C}$. Given $\mathbb{K}, \mathbb{L} \in \mathfrak{K}$ with $\mathbb{K} \subset \mathbb{L}$ than automorphisms of the latter induce the identity on the former. That is, every restriction $\rho_{\mathbb{L},\mathbb{K}} : Gal(\mathbb{L} : \mathbb{Q}) \to Gal(\mathbb{K} : \mathbb{Q})$ is a homomorphism an even an epismorphism since it can be extended to the hole space.

The restriction maps and the Galois groups of the elements of $\mathfrak{K}$ form a projective system and

$$Gal(\bar{\mathbb{Q}}/\mathbb{Q}) = \lim_{\leftarrow} Gal(\mathbb{K} : \mathbb{Q})$$

$Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is called a profinite group.[1]So, some properties of Galois theory can be extended but still its structure is much less known.

EXAMPLE. $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is not finitely generated. Indeed,

[1]Term introduced by Serre in the fifties, coming from <u>pro</u>jective limit of <u>finite</u> groups.

$$\# \lim_{\leftarrow} Gal(\mathbb{K} : \mathbb{Q}) = 2^{\aleph_0}$$

- Now we state a short theorem, from last century, still very usefull in algebraic geometry.

DEFINITION 3.1.1. Let $\mathbb{L}$ be a unique factorisation domain. Let $f, g \in \mathbb{L}[X]$ be polynomials with

$$\begin{array}{rcl} f(x) & = & \sum_{i=0}^{n} \phi_i x^i \\ g(x) & = & \sum_{j=0}^{m} \gamma_j x^j \end{array}$$

where we suppose $\phi_n \neq 0 \neq \gamma_m$.

we define the resultant of f and g, $Res(f, g)$ as

$$Res(f,g) = \begin{vmatrix} \phi_n & \cdots & \cdots & \phi_0 & 0 & 0 \\ 0 & \ddots & & & \ddots & 0 \\ 0 & 0 & \phi_n & \cdots & \cdots & \phi_0 \\ \gamma_m & \cdots & \cdots & \gamma_0 & 0 & 0 \\ 0 & \ddots & & & \ddots & 0 \\ 0 & 0 & \gamma_m & \cdots & \cdots & \gamma_0 \end{vmatrix}$$

THEOREM 3.1.2. *(of the resultant) Given the notations above, f and g have a non constant common factor if and only if Res(f,g)=0.*

We need a small lemma:

LEMMA 3.1.3. *Given the notations above f and g have a non constant factor in common if and only if there exists polynomials* $\Phi\,\Gamma \in \mathbb{L}[X]$ *such that*

$$\begin{array}{ccccc} 0 & < & deg(\Phi) & < & deg(f) \\ 0 & < & deg(\Gamma) & < & deg(g) \\ & f.\Gamma & = & g.\Phi & \end{array}$$

PROOF. (of lemma) We suppose none of our polynomials divides the other since then the result is obvious.

Let's take h the common factor to f, and g; we define Φ and Γ as $g = h.\Gamma$, $f = \Phi.h$.

On the other side if we have $f.\Gamma = g.\Phi$ each irreducible factor of g divides the left-hand side, since the degree of Γ is smaller than that of g there must be must be at least one that divides f. □

We can now prove the Resultant theorem.

PROOF. (of theorem) We rewrite Φ and Γ in a sum like we did for f and g and we note explicitly the equation $f.\Gamma = g.\Phi$. This gives us a homogenous system with a non-zero solution. Putting this system into a matrix, we see that it is Res(f,g) and so having a non trivial solution; by Cramer it's determinant must be zero and it's done. □

3.2. The Belyi Theorem

The history of Finland and Russia have been bounded for ages, so it is not a surprise that the key theorem that Grothendieck required to confirm his intuition of the role of dessins was presented by a Russian mathematician (Belyi) in the Helsinki international mathematical congress. The simplicity of the proof contrasts with the deepness of the statement. Deligne who attended the congress sent it to Grothendieck on the back of a letter.

THEOREM 3.2.1. *(Belyi - 1979) Let X be an algebraic function defined over $\mathbb{C}$ then X is defined over $\overline{\mathbb{Q}}$ if and only if there exist a holomorphic function $f : X \to \mathbb{P}^1\mathbb{C}$ such that all critical values lie in* $\{0, 1, \infty\}$.

PROOF. The if part is proved by using a Weil's rigidity criterion.[2]Let's go to the only if...

Suppose X is defined over $\overline{\mathbb{Q}}$ and let $g : X \to \mathbb{P}^1\mathbb{C}$ be s.t. all it's critical values lie in $\overline{\mathbb{Q}}$. We construct a morphism $h : X \to \mathbb{P}^1\mathbb{C}$ all of whose critical values lie in $\mathbb{Q}$. We define S as the set of critical values of g and all their $Gal(\overline{\mathbb{Q}}/Q)$ conjugates. We set $f_0(z_0) = \Pi_{s\in S}(z_0 - s)$, $f \in \mathbb{Q}[z_0]$ and we set

$$f_{j+1}(z_{j+1}) = Res_{z_j}\left(\frac{df_j}{dz_j}, f_j(z_j) - z_{j+1}\right)$$

So, the roots of f_{j+1} are the finite critical values of f_j. These are defined over the rational numbers and their degree decrease as their index increase. So there must be a n for which $deg(f_n) = 0$. We define h as

$$h = \bigcirc_{i=0}^{n-1} f_i \circ g$$

The critical values lie in $\mathbb{Q}$ since the critical values of $l \circ m$ are the critical values of l union with the image by l of the critical values of m. We denote by S' the set of finite criticical values of h.

[2]I won't give a proof here since it is long, quite hard but still 'automatic'. For more detail see Wolfart's article **[23]**.

We now work by iteration on the number of elements of S'. If $\#S' \leq 3$ a single linear fractional transformation can send the elements of S' onto a subset of $\{0, 1, \infty\}$. If $\#S' > 3$ we chose three ordered elements that by a linear fractional transformation we send onto $0, m/(m+n)$ and 1. Then applying the transformation

$$z \mapsto \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$$

sends them respectively to 0,1 and 0. So we obtain by composition a morphis such that it's set of cardinal values strictly decreases. We repeat the operation till we have a critical value set of strictly less then four elements. □

DEFINITION 3.2.2. A *Belyi morphism* is a morphism $\beta\, X \to \mathbb{P}^1\mathbb{C}$ who's critical values are elements of $\{0, 1, \infty\}$. A *pre-clean Belyi morphism* is a Belyi morphism where all the ramifications order above 1 are less or equal to 2. A *clean Belyi morphism* is a Belyi morphism where all the ramifications above 1 are strictly equal to 2.

The pre-clean and clean term are not there by mistake as we can already guess , their relation with their homonym dessins adjectives will appear soon.

DEFINITION 3.2.3. A *Belyi pair* is an algebraic curve defined over $\overline{\mathbb{Q}}$ and a Belyi morphism defined on it. Two Belyi pairs $(X, \phi); (Y, \psi)$ are isomorphic if there exist an isomorphism $\mu : X \to Y$ such that $\psi = \mu \circ \phi$.

FACT 3.2.4. *An algebraic curve defined over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if and only if there exists a clean Belyi morphism $\gamma\, X \to \mathbb{P}^1\mathbb{C}$.*

PROOF. It is a corollary of the last theorem and of the obvious fact that if μ is a Beliy morphism then $4\mu(1-\mu)$ is clean. □

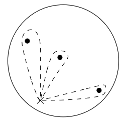
To conclude we need this classical lemma.

LEMMA 3.2.5. *There's a bijection between the conjugacy classes of finite index subgroups of $\pi_1(\mathbb{P}^1\mathbb{C} \setminus \{0, 1, \infty\})$ and the isomorphism classes of finite coverings of $\mathbb{P}^1\mathbb{C}$ only ramified above 0,1 and ∞.*

Finally,

Alternative 4

THEOREM 3.2.6. *There exists a bijection between the set of isomorphism classes of clean Beliy pairs and the set of clean dessins.*



PROOF. Let π_1 denote the Poincaré group of $\mathbb{P}^1\mathbb{C}$. π_1 is generated by $< l_0, l_1, l_\infty >$ that are the loop respectively around 0, 1 and ∞.With the relation $l_0 l_1 l_\infty = 1$. Let us remember the definition of $\mathfrak{C}_2^+$:

$$\mathfrak{C}_2^+ =< \rho_0, \rho_1, \rho_2 \left| \rho_1^2 = \rho_0\rho_1\rho_2 = 1 \right. >$$

On the other side let's have a look at $\pi_1/ < l_1^2 >$. The quotient only affects the words including l_1. Specifically $l_1^2 = 1$. $l_0 l_1 l_\infty$ remains the same.

In fact these two groups are isomorphic. So by the last lemma and theorem we conclude. □

We have proved the main correspondance relation concerning dessins d'enfants. We won't give any concrete examples here since it is the aim of the next part where this *Grothendieck correspondance* is studied more deeply. We will also see why it was interesting for the study of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

We already see that to poles above 0,1 and ∞ correspond the elements of X_0,$X_1 \setminus X_0$ and $X_2 \setminus X_1$ respectively.

To conclude this first part here is a short synthesis of the different faces of dessins d'enfants (non-exhaustive).

Synthesis of the different faces.

Dessins d'enfants are :

1. Grothendieck dessins d'enfants - bipartite graphs embedded into compact oriented surfaces.
2. Tripartite triangulations of the same surfaces.
3. Bicolored triangulations of the same surfaces.
4. Finite index subgroups of the oriented cartographic group $\mathfrak{C}_2^+$.
5. Belyi functions defined on these surfaces - functions defined over $\mathbb{P}^1\mathbb{C}$ with at most three critical values being 0,1 or ∞.

Most of thes equivalences were known before, the great new one is number 5 because it permits us to study the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ indeed ...

Part 2

The case of plane trees and/in the use of dessins in the study of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

CHAPTER 4

Action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on plane trees.

4.1. Introduction

The interest in introducing dessins in the study of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) is that, as we will see, the action of our group on dessins is faithful. Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) can so be studied by observing how it acts on 'trivial' objects. In this Part we'll specifically work out the action on trees.

4.2. Definitions

DEFINITION 4.2.1. A *tree* is a genus 0 dessin $X_0 \subset X_1 \subset X_2$ where $X_2 \setminus X_1$ consists of a single open cell.

This definition corresponds to the heuristic definition of a tree. An example is the Leila flower we saw before. (In fact I should say Leila's flower*s* since all the trees, consisting of all permutation of the final clusters bearing two, three, four and five leaves, are called like that)

DEFINITION 4.2.2. The action ϕ of a group G on a set X is *faithful* if $\ker \phi = \{e\}$ where e is the identity of G.

EXAMPLE 4.2.3. Let's take $X = \mathbb{N}$, $G = S_2$.

If we take $\phi_1 = \{((12), x); ((2,1), x)\}$ then ϕ_1 is effectively a homomorphism but isn't very interesting for the study of G.

If we take $\phi_2 = \{((12), x); ((21), x+1-2(x \mod 2)\}$ that is faithful, the study of the composition of our two functions tells us everything about G.

These two extreme examples show us the importance of being faithful.

4.3. Faithfullness of the action on trees

THEOREM 4.3.1. *(Lenstra Jr.) The action of Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) on the set of trees is faithful.*

To show this we need two small lemma's.

LEMMA 4.3.2. *Let F be a polynomial of degree n and let d divide n. If there exists a polynomial H monic, of degree d, with no constant term such that for a polynomial G we have $F = G \circ H$ then H is unique.*

PROOF. We can write F down as

$$F = \sum_{i=0}^{n} \phi_i x^i$$

with $\phi_n \neq 0$.

Let the degree of G be m ($n = m.d$). We write G and H respectively as

$$\begin{aligned} G &= \gamma_m x^m + \cdots + \gamma_0 \\ H &= x^d + h_{d-1}x^{d-1} + \cdots + h_1 x \end{aligned}$$

We can thus rewrite F as

$$F = \gamma_m H^m + \cdots + \gamma_0$$

Note that $\phi_0 = \gamma_0$ and $\phi_n = \gamma_m$. We can notice that the $m.d-(m-1).d = d$ highest terms coefficients of F are only defined by polynomials in h_i, G has 'nothing to say'. To show that $h_d, \cdots, h_1$ are well and uniquely defined by this system (the one induced by comparing the d highest terms) we must only solve it inductively starting with h_{d-1} and we will always have the h_i as the unique solution of a linear system. Not forgetting $h_0 = 0$, the lemma is proved. □

LEMMA 4.3.3. *Let $G, H, \bar{G}$ and $\bar{H}$ be polynomials such that $G \circ H = \bar{G} \circ \bar{H}$ and $deg(H) = deg(\bar{H}) = d$. Then $\bar{H}$ is the image of H by a affine polynomial.*

PROOF. Let h_d (resp. $\bar{h}_d$) be the leading coefficient of H(resp.$\bar{H}$) and h'_0 (resp.$\bar{h}'_0$) be the constant coefficient of H/h_d (resp.$\bar{H}/\bar{h}_d$). There exist polynomials $G', \bar{G}'$ such that $G' \circ (H/h_d - h'_0) = \bar{G}' \circ (\bar{H}/\bar{h}_d - \bar{h}'_0)$. These are just affine transforms of G and $\bar{G}$ respectively.

Both $(H/h_d - h'_0), (\bar{H}/\bar{h}_d - \bar{h}'_0)$ are monic, of same degree, with no constant term. We can apply previous lemma and find

$$\frac{H}{h_d} - h'_0 = \frac{\bar{H}}{\bar{h}_d} - \bar{h}'_0$$

that is

$$\frac{\bar{h}_d}{h_d} H - \bar{h}_d.(h'_0 - \bar{h}'_0) = \bar{H}$$

□

We can now jump to the proof of the theorem.

PROOF. *(of the theorem)*

Let $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. We will a create a tree D such that the action of σ on it is non-trivial. Let $\mathbb{K}$ be a number field and α a primitive element for $\mathbb{K}$ such that σ acts non-trivially on $\alpha(1)$. Instead of creating a topoligical $X_0; X_1; X_2$ tree we will (by previous chapter) create a Beliy function $\beta(z)$ defined over $\mathbb{K}$ such that $\beta(\frac{az+b}{cz+d}) \neq \beta^\sigma(z)$[1]except when $\frac{az+b}{cz+d} = z$ (i.e. the action of σ on our dessin is not just an automorphism). Our rational beliy function is supposed to be a tree, so there's one open cell on the spere that is our function β is a polynomial. The unique point over ∞ is ∞.

If a polynomial β is such that $\beta^\sigma(z) = \beta(\frac{az+b}{cz+d})$ then c=0 and one can set d to 1 (replacing our a,b by multiples doesn't change anything). We will thus create β a belyi polynomial such that $\beta^\sigma(z) = \beta(az+b)$ implies that $az+b$ is the identity on $\mathbb{C}$.

We define $f_\alpha(z) \in \mathbb{K}[z]$ as being the polynomial whose derivative is

$$f'_\alpha(z) := z^3(z-1)^2(z-\alpha)$$

By the proof of Belyi's theorem there exists $f \in \mathbb{Q}[z]$ such that $f \circ f_\alpha$ is a Beliy polynomial. We call it g_α. Let $\mu := \alpha^\sigma$, $\mu \neq \alpha$ by (1). Since f is defined over $\mathbb{Q}$ (and $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ fixes $\mathbb{Q}$) we have that $g_\mu := f \circ f_\mu$, where $f_\mu = f_\alpha^\sigma$, is also a belyi polynomial. Let T_α (resp. T_μ) be the tree corresponding to g_α (resp. g_μ) ($T_\mu = T_\alpha^\sigma$), we'll show that these two trees are distinct. That is $g_\mu(z) \neq g_\alpha(az+b)$ if $(a,b) \neq (1,0)$.

Suppose we have (a,b) such that $g_\mu(z) = g_\alpha(az+b)$. That is

$$f \circ f_\mu(z) = f \circ f_\alpha(az+b)$$

Applying previous lemma we have that $f_\alpha(az+b) = e.f_\mu(z)+f$ for some constants e,f. Let's have a look at the critical points of both these functions. The right term function has the same critical points as f_μ. The left term function has the pre-image of 0,1 and α under $az+b$ as critical points. In short, with multiplicities

$$\begin{array}{lcccc} l.t.f. & : & -\frac{b}{a}, 3 & \frac{1-b}{a}, 2 & \frac{\alpha-b}{a}, 1 \\ r.t.f. & : & 0, 3 & 1, 2 & \mu\, 1 \end{array}$$

Identifying the two terms, and comparing the two first critical loci we get $(a,b) = (1,0)$ and that gives us for the last critical point $\mu = \alpha$ and that

[1]We denote by β^σ the image of β under the action of σ.

is contrary to our basic assumption. So $g_\mu(z) = g_\alpha(az+b)$ can never be true with a and b respectively different from 1 and 0. □

This important result shows us that studying the action of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ on plane trees can also give us non-trivial information. This is quiet interesting since trees are a subset of dessins composed of relatively simple elements. Our theorem also has an immediate corollary.

COROLLARY 4.3.4. *$Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of genus 0 dessins.*

Before going further in the study of the Grothendieck correspondance we will work some simple examples out explicitly. This simple basis will be useful later in visualising what is happening with more intricate examples or inside more abstract proofs.

DEFINITION 4.3.5. The *valency* of a vertex is the number of edges starting from it. It is a local property and a loop is thus counted twice. The *valency* of an open cell is the number of edges on its boundary (an edge being counted twice if bounded on both sides by the cell).

REMARK 4.3.6. Dessins on a fixed surface are sometimes defined by their valencies (of vertices), this is not always enough since different graphs can have the same valencies (e.g. a three branch tree with resp. one, two and three leaves, modifying the order of appearance of leaves (S_3 action) can change the dessin)

EXAMPLE 4.3.7. $\beta(z) = 4.z^k.(1-z^k)$ with $k \in \mathbb{N}_0$.
$\beta^{-1}(0) = \{0, \zeta_1^k, ..., \zeta_k^k\}$ [2]with multiplicities (> valencies) k,1...1.
$\beta^{-1}(1) =?$

$$1 = 4.z^k.(1-z^k)$$

$$4.z^{2k} - 4.z^k + 1 = 0$$

this a bisquared polynomial:

$$4.y^2 - 4.y + 1 = 0 \wedge z^k = y$$

[2]The ζ^k's are the k-th roots of unity.

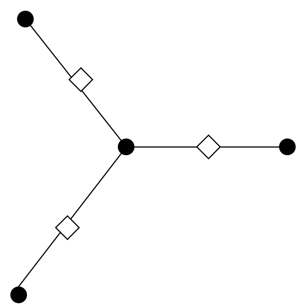
$$y = \frac{1}{2} \wedge z = \{\frac{\zeta_i^k}{2^{1/k}}, i = 1...k\} = \beta^{-1}(1)$$

the multiplicities are all 2.

$\beta^{-1}(\infty) = \infty$ with multiplicty 2k.

We see that T_β dessin corresponding to β is a tree : it is a polynomial. If we notice that our tree is simply connected then there's only one way (up to automorphism) to connect our points and we see that it is a clean tree (star) with k branches. If we don't notice it, we can just use the symmetry of β under multiplication by ζ_1^k and look along the x-axis to obtain the same result with the plus-value that we know that our tree is here (in this form) made of perfecty straight edges.

In fact this result could have been found much faster using preliminary results... indeed, β is the doubling of z^k... and with this function, the form of the tree is obvious!



We take now another easy example that is not a tree.

EXAMPLE 4.3.8. $\beta(z) = -\frac{(z^n-1)^2}{4.z^n}$ with $n \in \mathbb{N}_0$

$\beta^{-1}(0) = \{\zeta_1^n, ..., \zeta_n^n\}$ each with multiplicity 2.

$\beta^{-1}(1) =?$

$$1 = -\frac{(z^n - 1)^2}{4.z^n}$$

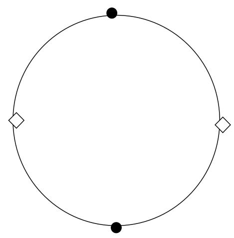
We set $y = z^n$ and we solve for y, this gives us

$$y = -1$$

The z are thus the opposites of the ζ^n also with multiplicity 2.

$\beta^{-1}(\infty) = \{0, \infty\}$ we see immediately that we don't have a tree anymore.

D_β is a dessin composed of two cells, with the same arguments as before we can see that it is a clean dessin corresponding to a circle with n points (alternated) of type $\star$ and $\bullet$.



4.4. Calculation of the beliy function in the genus 0 case.

The examples we just worked out were in the direction Beliy function -> Dessin. They were quiet easy to calculate. In general, calculating the pre-image of [0,1] isn't that easy, specially round the ramification points. Different techniques exist to help us draw the dessin rigorously. The other way round is also often quiet intrincate since big numbers are used in important polynomial systems. The calculation of the Grothendieck correspondance is however often important because it gives explicit examples on which ideas can be tried.

We will start here by studying the Dessin -> Beliy function technique. A first remark is that our function will be defined up to composition with an element of $SL_2(\mathbb{C})$ the group of automorphisms of $\mathbb{P}^1\mathbb{C}$. We will first see the process to follow to go from a dessin to a beliy function, the justifications will follow.

4.4.1. Characterisation of a dessin by means of valencies. Let's take a dessin D, we will suppose D is clean (each edge is bounded by a $\bullet$). To our dessin we will associate two lists of valencies. We set $n = n_D$ and $m = m_D$ the maximum valency of the vertices and cells of D.

$$D \mapsto \begin{cases} V = [v_1, ..., v_n] \\ C = [c_1, ..., c_m] \end{cases}$$

where v_i (resp. c_i) is the number of vertices (resp. cells) of valency i. The number e of edges of D is, by Euler's formula : $e = \frac{1}{2}.\{\sum_i v_i + \sum_j c_j - 2\}$.

4.4.2. Definition of the $\tilde{R}$ system of equations. We set

$$\begin{aligned}\tilde{P}_i(z) &= z^{v_i} + \textstyle\sum_{k=0}^{v_i-1} C_{i,k}.z^k \quad (1 \leq i \leq n)\\ \tilde{Q}_j(z) &= z^{c_j} + \textstyle\sum_{l=0}^{c_l-1} D_{j,l}.z^l \quad (1 \leq j \leq m)\end{aligned}$$

where the $C_{i,k}; D_{j,l}$ are the indeterminates. We denote by $\tilde{R}_{V,C}$ the system consisting of $\{\tilde{P}_i; \tilde{Q}_j\}$; $\tilde{R}_{V,C}$ doesn't refer to a single dessin but to a family a dessins, these defined up to automorphisms of the surface. We will show that our system in $C_{i,k}; D_{j,l}$ has an algebraic solution such that when substituted into the polynomials we have

$$\beta(z) := \frac{\prod_{i=1}^{n} \tilde{P}_i(z)^i}{\prod_{j=1}^{m} \tilde{Q}_j(z)^j}$$

that becomes a beliy function associated to our original dessin.

4.4.3. Definition of the R system of equation. Our β will also be defined up to automorphism, to fix it we will arbitrarly fix three indeterminates. We'll go from a system $\tilde{R}_{V,C}$ to a system $R_{V,C}$ where the $SL_2(\mathbb{C})$ action will be fixed.

4.4.3.1. *Step 1.*

NOTE. Here we send the center of cell or a vertex to infinity.

1. If the dessin has a single vertex (of valency i_0), i_0 must be strictly greater than 1 because we have assumed that D was clean. We set

$$P_{i_0}(z) = \tilde{P}_{i_0}(z) - (C_{i_0,1} + 1).z - C_{i_0,0}$$

2. If the dessin has more than one vertex we will take a $j_0 \in \{1, ..., m\}$ or $i_0 \in \{1, ..., n\}$ in order that v_{i_0} or c_{j_0} is a minimum in $I = \{u_i : 1 \leq i \leq n, u_i \neq 0\} \cup \{c_j : 1 \leq j \leq m, c_j \neq 0\}$. If that is a i_0 we put

$$P_{i_0}(z) = \alpha(z^{v_{i_0}-1} + C_{i_0,v_{i_0}-2}.z^{v_{i_0}-2} + ... + C_{i_0,1}.z + C_{i_0,0})$$

If that is a j_0 then

$$Q_{j_0}(z) = \alpha(z^{c_{j_0}-1} + D_{j_0,c_{j_0}-2}.z^{c_{j_0}-2} + ... + D_{j_0,1}.z + D_{j_0,0})$$

where the α is an indeterminate.

4.4.3.2. *Step 2.*

NOTE. Here we fix a $C_{i,k}$ or $D_{j,l}$ to the values 0 or 1 (in bijection).

1. If the dessin has a single vertex than, by cleanness it consist of at least two open cells since the dessin must consist of closed loops. We set

$$Q_1(z) = \tilde{Q}_1(z) - (D_{1,1} + 1).z - D_{1,0}$$

2. If the dessin has more than one vertex:

$$if\ \exists i_1 \in \{1, ..., n\} | v_{i_1} > 1 \wedge (i_0\ defined \Rightarrow i_1 \neq i_0)$$

then set

$$P_{i_1}(z) = \tilde{P}_{i_1}(z) - (C_{i_1,1} + 1).z - C_{i_1,0}$$

on the other hand

$$if\ \exists j_1 \in \{1, ..., m\} | c_{i_1} > 1 \wedge (j_0\ defined \Rightarrow j_1 \neq j_0)$$

then set

$$Q_{j_1}(z) = \tilde{Q}_{j_1}(z) - (D_{j_1,1} + 1).z - D_{j_1,0}$$

If all the v_i and c_j different from zero are equal to one (except the i_0 or j_0) then we can always choose a couple of this form; (a) (i_1, i_2); (b) (i_1, j_1); (c) (j_1, j_2); where if i_0 was chosen (resp. j_0) than $i_1, i_2 \neq i_0$ (resp. $j_1, j_2 \neq j_3$. Depending on the type we now set

$$(a) \rightsquigarrow \begin{cases} P_{i_1}(z) = \tilde{P}_{i_1}(z) - C_{i_1,0} \\ P_{i_2}(z) = \tilde{P}_{i_2}(z) - C_{i_2,0} + 1 \end{cases}$$
$$(b) \rightsquigarrow \begin{cases} P_{i_1}(z) = \tilde{P}_{i_1}(z) - C_{i_1,0} \\ Q_{j_1}(z) = \tilde{Q}_{j_1}(z) - D_{j_1,0} + 1 \end{cases}$$
$$(c) \rightsquigarrow \begin{cases} Q_{j_1}(z) = \tilde{Q}_{j_1}(z) - D_{j_1,0} \\ Q_{j_2}(z) = \tilde{Q}_{j_2}(z) - D_{j_2,0} + 1 \end{cases}$$

4.4.3.3. *Step 3.*

NOTE. Here we just leave the other polynomials unchanged. It is just taxonomy.

For all i, j respectively elements of $\{1, ..., n\}$ and $\{1, ..., m\}$ that were not defined previously in 1. and 2. we set $\begin{matrix} P_i(z) = \tilde{P}_i(z) \\ Q_j(z) = \tilde{Q}_j(z) \end{matrix}$ and our system $R_{V,C}$ is defined by $\{P_i; Q_j\}$ for all indices.

THEOREM 4.4.1. *Given D a genus 0 dessin, D having a vertex at the end of each edge. Let* $V = \{v_1, ...v_n\}$ *and* $C = \{c_1, ..., c_m\}$ *be the vertex and cell valency lists of D; let* $R_{V,C}$ *be the system of polynomial* $\{P_i; Q_j\}$ *defined as* *above; let e be the number of edges of D; let* $B_0, ..., B_{e-1}$ *be indeterminates; let* $B(z) := z^e + \sum_{i=0}^{e-1} B_i.z^i$*; let* $A(z) = \prod_{i=1}^{n} P_i(z)^i$*; let* $C(z) = \prod_{i=1}^{m} Q_i(z)^i$*. We define the system* $S_{V,C}$ *as the system obtained by comparing term by term the coefficients of the equation*

$$A(z) - C(z) = \pm B(z)^2$$

where the sign attributed to the right is plus if deg(A)>deg(C) and minus if deg(C)>deg(A)[3]*. Note that S is a system with 2e equations and as many undeterminates. We have*

(1) For each solution s of the system $S_{V,C}$ *we have* $\beta_s(z) := A(z)/B(z)$*, where the coefficients of A and B have been replaced by the solutions of S corresponding to s, that is a clean beliy function.*

(2) The dessins corresponding to the β_s *are those with valencies V, C. There exists at least one solution s such that* $D = \beta_s^{-1}([0; 1])$*.*

(3) The system $S_{V,C}$ *admits only a finite number of solutions s. They are defined over* $\bar{\mathbb{Q}}$ *and so the same holds for the* $\beta_s(z)$*.*

We need a lemma for the proof of (1)

LEMMA 4.4.2. *(i) Let A(z), B(z) and C(z) be polynomials in* $\mathbb{C}[z]$*. Suppose that B(z) has distinct roots and that* $A(z) - C(z) = B(z)^2$*. For a polynomial* $D(z) = \prod_i (z - a_i)^{n_i}$ *we denote* $\tilde{D}(z) = \prod_i (z - a_i)^{n_i - 1}$*. Suppose finally that* $A.C' - C.A' = \tilde{A}.\tilde{C}.B$*. Then* $\beta(z) = A(z)/C(z)$ *is a clean belyi function.*

(ii) On the other side, if $\beta(z) = A(z)/C(z)$ *is a clean belyi function then* $A(z) - C(z) = B(z)^2$ *where B(z) is a polynomial in* $\mathbb{C}[z]$ *having distinct roots.*

PROOF. (of lemma)

(ii) Let $\beta(z) = A(z)/C(z)$ be a clean belyi function, $\beta(z) - 1 = A(z)/C(z) - 1 = (A(z) - C(z))/C(z)$ has roots of order two. So A(z)-C(z) must be equal to the square of polynomial B(z) having distinct roots.

(i) Let $\beta(z) = A(z)/C(z)$ where A and C are those from the hypothesis. We have

$$\beta'(z) = \frac{A'(z).C(z) - A(z).C'(z)}{C(z)^2}$$

That is (by hypothesis)

$$\beta'(z) = \frac{\tilde{A}(z).\tilde{C}(z).B(z)}{C(z)^2}$$

[3] note that A and C are never of same degree

The roots of β' are the multiple roots of A and C (tilding reduces the multiplicity of roots by one) and the roots of B (that are simple by hypothesis). On a root of A, $\beta \to 0$; on a root of C, $\beta \to \infty$ and on a root of B $\beta \to 1$. So we have β that is a beliy function to see that it is clean, one may notice that B has simple roots and so the ramification indices over 1 are always 2. □

We can now prove our result.

Proof. (of theorem)

(1) is a corollary of the last lemma.

(2) corollary of the last lemma and the fact that the valencies correspond to the order of ramifications above the cells and vertices (∞,0 and 1)

(3) Suppose $S_{V,C}$ has infinitely many solutions s. Specifically, given a dessin D, there exists a dessin D' with same valencies V,C such that an infinite number of solutions s induces beliy functions β_s corresponding to D'. By the construction of our system $R_{V,C}$ we have either a vertex of D' of valency i_0 or an open cell of valency j_0 at ∞ and a vertex of valency i_1 or an open cell of valency j_1 must be at 0. The condition $C_{i_1,1} = 0$ means that the product of the vertices of valency i_1 is one. There are only a finite number of realisations of D' as the pre-image of a rational beliy functions satisfying these conditions. In particular one of this representations is given by an infinite number of beliy functions β_s; that is impossible by the Grothendieck correspondance. □

This algorithm to render a polynomial given a dessin is far from being optimized. Optimized: concerning the field of definition of the polynomial, we do not always get the smallest. Optimized: on speed; it is not always the fastest. Different (but fairly similar) algorithm exist to solve this problem. The interested reader can read Granboulan and Couveignes or Wolfart. Nevertheless this algorithm has a good point: used with Groebner basis algorithm it doesn't need any human action once programmed. We won't use it here since it generates big systems of equations and even big solutions if the dessin was taken too complicated at the beginning.

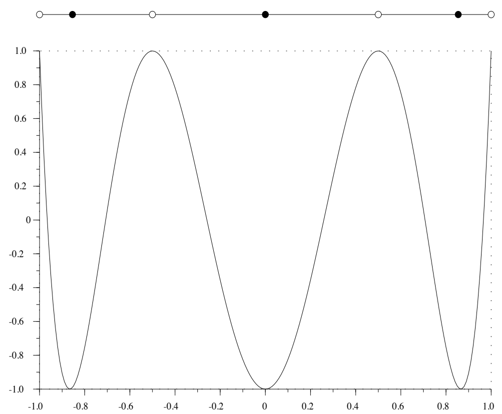
We will now have a look at the orbit of a dessin. We will work with trees.

4.5. Trees.

We start with a simple example that will let us guess the way trees were studied.

EXAMPLE 4.5.1. *(linear* trees*) We define* $\beta(z) = \Theta^{-1} \circ T_d \circ \Theta$ where $T_d : x \mapsto \cos(d.\arccos(x))$ is the d-th Tchebychev polynomial and where $\Theta\, x \mapsto 1 - 2.x$.

We have Θ that sends 0 and 1 respectively on 1 and -1. Remembering that the only critical values of the Tchebychev polynomial are ± 1 we find immediatly (basic trigonometry) that our T_β corresponds to linear trees (clean or unclean depending on the parity of β). We see here a Tchebychev polynomial with domain restricted to $\mathbb{R}$ before being 'acted' by Θ and the corresponding tree.



Tchebychev polynomials generalised in the way we will see, will indeed permit us to generate all plane trees.

DEFINITION 4.5.2. A *generalized Tchebychev polynomial* is a non-constant polynomial over an algebraic closed field k with no more than two critical values. In extenso, there are values $\{\omega_1, \omega_2\} \subset k$ such that if $P'(a) = 0$ then $P(a) = \omega_{1or2}$. If these critical values are ± 1 we say that it is *normalized.*

EXAMPLE 4.5.3. The star we have studied before (original z^k or doubling) are polynomials with no more than two critical values. They are thus generalized Tchebychev polynomials.

PROPOSITION 4.5.4. *We have;*

(i) Let P(z) be a normalized Tchebychev polynomial, and let $\beta(z) := 1 - P(z)^2$*. Then,* β *is a clean beliy polynomial and* $\beta^{-1}([0,1])$ *is a tree with the* ∞ *inside its open cell.*

(ii) Let T be a tree. Then there is a normalised generalised Tchebychev polynomial P, such that with $\beta := 1 - P^2$ *we have T=*$\beta^{-1}([0,1]\}$

PROOF. (i) $\beta(z) = 1 - P(z)^2$ has only 0 and 1 as critical values since P is normalised. It is thus a Beliy function. It has only one pole and so the pre-image of $[0, 1]$ is a tree.

(ii) Given T a tree there's a rational beliy function β such that the pre-image under beta of [0,1] is T. T is a tree so *beta* has only one pole. As done earlier, by composing by automorphisms ($\in SL_2(\mathbb{C})$ we have beta a Beliy polynomial with critical values 0 and 1. By cleannes we must have $\beta(z) - 1 = c.Q(z)^2$ with c a constant and Q a polynomial with distinct roots. Critical points of β are the critical points and roots of Q. β's critical values may only be 0 or 1 and 1 at a root. So there exists a critical point $\tilde{z}$ of Q which is not a root at which

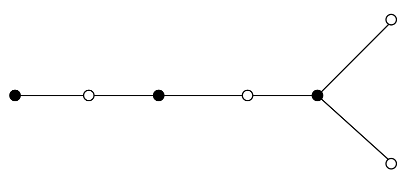
$$1 + c.Q(\tilde{z})^2 = 0$$

that is $Q(\tilde{z}) = \pm\sqrt{-1/c}$. If we define $P(z) = \sqrt{-c}.Q(z)$ then we have the required polynomial. (critical values of P are ± 1. □

We describe here another algorithm to generate beliy functions starting from a tree. Indeed, we could use the algorithm seen before but given the simpler structure of trees , our algorithm will also be simpler.

DEFINITION 4.5.5. Given a bipartite structure S on our tree T with images in $\{\pm 1\}$, we denote the valency of vertex positive (resp. negative) if the image of the vertex by S is $+1$ (resp. -1). The *positive valency list* of a tree (resp. *negative valency list*) is a list of length m_+(resp. m_-), that is the maximum positive (resp. negative) valency of vertex. It is defined as $V_\pm = [v_{\pm,1}, ..., v_{\pm,m_\pm}]$ where $v_{\pm i}$ is the number of vertices of pos./neg. valency i.

EXAMPLE 4.5.6. For the tree we will study later:



We have : $V_+ = [2, 2]$ and $V_- = [1, 1, 1]$.

REMARK. These valencies are defined uniquely up to inversion of S.

ALGORITHM 4.5.7. (Definition of beliy functions given the positive and negative valency lists of a tree)

The construction is similar to that of general dessins. Given a tree T *and* $V_+ = [v_{0,+}, ..., v_{m,+}]$, $V_- = [v_{0,-}, ..., v_{n,-}]$ *its* $\pm$*valency lists. We set*

$$\tilde{P}_i(z) = z^{v_{i,+}} + \sum_{l=0}^{v_{i,+}-1} C_{i,l}.z^l \; (1 \leq i \leq m)$$

$$\tilde{Q}_j(z) = z^{v_{j,-}} + \sum_{l=0}^{v_{j,-}-1} D_{j,l}.z^l \; (1 \leq j \leq n)$$

where the $C_{i,j}; D_{i,j}$ *are indeterminates. Define* $\tilde{R}_{V_+,V_-}$ *the set of polynomials* $\{\tilde{P}_i, \tilde{Q}_j\}$. $\tilde{R}$ *depends only on the* $\pm$*valency lists and so applies only to a finite number of trees. As before we transform our set* $\tilde{R}$ *into a set* R*. We do as follows:*

We choose $i_0 \in \{1, ..., m\}$ *such that* $v_{i_0,+} \neq 0$ *and we set*

$$P_{i_0}(z) = \tilde{P}_{i_0} - C_{i_0,0} - (C_{i_1,1} + 1).z$$

For all $i \neq i_0$ *we set* $P_i(z) = \tilde{P}_i(z)$*. We leave the* Q *unchanged (they just lose their tilde).* R *is defined as the set of* $\{P_i, Q_j\}$*. We have the following theorem:*

THEOREM 4.5.8. *Let* T *be a tree, assumed to have a vertex at the end of each edge, with bipartite structure. Let* $V_\pm = \{v_{1,\pm}, ..., v_{m/n,\pm}\}$ *be its positive and negative valency list. Define*

$$P(z) = \prod_{j=1}^{n} Q_j(z)^j$$

and let S_{V_+,V_-} *be the set of polynomial equations obtained by comparing coefficients on both sides of*

$$P(z) - P(0) = \prod_{i=1}^{m} P_i(z)^i$$

We have:

(i) For each solution s *of* S_{V_+,V_-}*, let* $P_s(z)$ *be the normalized generalized Tchebychev polynomial given by replacing the indeterminates in the polynomial* $\frac{2}{P(0)}P(z) - 1$ *by the values of* s*, and let* $\beta_s(z)$ *be the polynomial obtained by replacing the indeteminates in the polynomial* $1 - P_s(z)^2$ *by the values of* s*. Then* $\beta_s(z)$ *is a clean beliy polynomial.*

(ii) The trees induced by $\beta_s(z)$ are exactly the set of trees of valency lists V_+, V_-.

(iii) The system S_{V_+,V_-} admits only a finite number of solutions, all defined over $\overline{\mathbb{Q}}$. In particular, all the $\beta_s(z)$ are defined over $\overline{\mathbb{Q}}$.

PROOF. (i) By construction P(z) is a generalized Tchebychev polynomial of critical values 0 and P(0). So $P_s(z)$ is a normalized generalised Tchebychev polynomial and by previous proposition it is a Beliy polynomial.

(ii) and (iii) are proved in a similar way as the (ii) and (iii) for the general case. □

4.6. Galois invariants.

4.6.1. What are these invariants? In order to study $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ by its action on trees we must be able to classify trees under their Galois orbits. Several invariants exist and one of the research topic is to fix combinatorial only invariants of dessins that would fix their orbit so that it wouldn't be necessary to calculate explicitly their associated Belyi morphism.

Here are some invariants:

4.6.1.1. *The number of faces, edges and vertices.*

4.6.1.2. *The valency list.* Here's the proof for these two first invariants in the case of the trees.

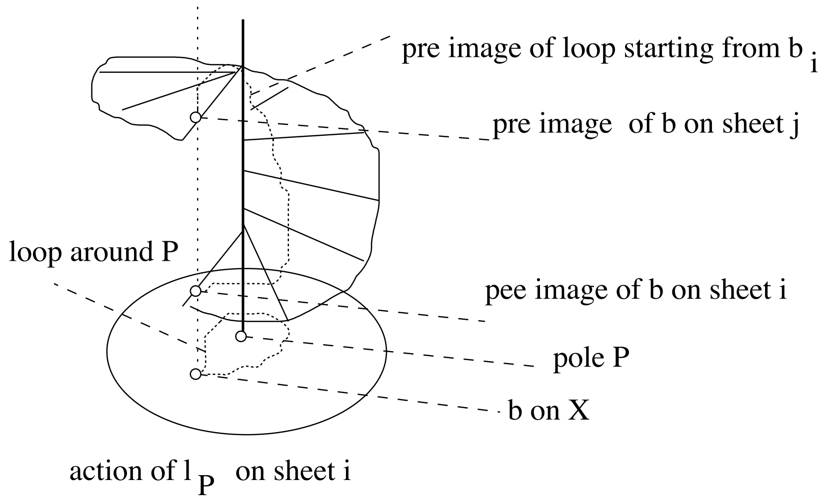
THEOREM 4.6.1. *The Galois orbit of a bicoloured plane tree lies in it's valency class (i.e. the trees with same ±valency lists.)*

PROOF. It comes from the fact that the valencies are the multiplicities of polynomial equations that are clearly Galois invariant (the Galois action on a tree is the action of the group on the roots of the polynomial defining the dessin).

Other invariants are: □

4.6.1.3. *The monodromy group of a dessin.* The monodromy of a dessin is, considering its Beliy morphism $\beta\ X \to \Sigma$, the *group* of permutation generated by $\pi_1(\Sigma)$ on the fibre of the basis point of a loop. In extenso:

It is the group whose element represent the lift by β of the different actions of π_1 on Σ. Visually the action is :



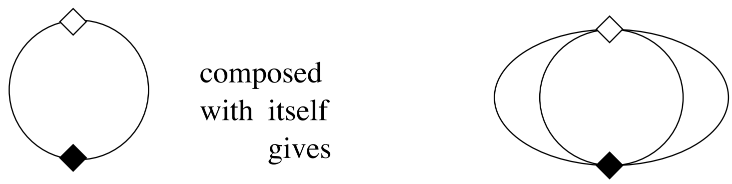
EXAMPLE. The monodromy of the n-star is C_n.

4.6.1.4. *The cartographical group of a dessin.* That is the subgroup of $\mathfrak{C}$ related to the dessin.

One can show (see for example [**14**]) that these groups are galois invariants. The second one being finer.

We also have a invariant related to compositions of dessins. Given three algebraic curves X, Y and Z and two ramified coverings $\beta : X \to Y; \gamma Y \to Z$ then $\beta \circ \gamma$ is the covering $X \to Z$ of Z.

If these coverings are dessins we say that this is a composition of dessins. A special case is the doubling we saw in chapter one. Here is another example



4.6.1.5. *So the galois orbit of a composition only contains dessins that are composed with the same basic dessin.*

4.6.2. What do we know about these invariants? We know from the work of Granboulan ([**10**]) That the composition and the valency invariants are necessary and sufficent conditions to separate orbits of Y-trees.

DEFINITION. A *Y-tree* is a tree where one and only one vertex has valency three and the others have valency 2 or 1.

EXAMPLE. The Coxeter graph of D_n with $n \geq 4$.

Usually for trees all the invariants above are sufficent to fix galois orbits. Usually, except in the case of e.g. Leila's flowers. For that case Zapponi introduced **another invariant**, similar to the signature of a permutation.

Some other interesting information we have, is about the the monodromy groups of trees. Indeed, we have the next theorem (for more information see Matzat) for meromorphic functions on compact Riemann surfaces with one pole - that is, trees, up to automorphism - that classifies the monodromy groups.

THEOREM 4.6.2. *Let G be the Monodromy group of the meromorphic function β defined on the compact Riemann surface X, with a single pole. Then the composition factors of G are in the following list:*

1. C_n, cyclic groups
2. A_n, alternating groups
3. M_{11}, M_{23}, two Mathieu groups
4. $PSL_n(\mathbb{F}_q)$, projective groups

PROOF. We have $\beta\ X \to \mathbb{P}^1\mathbb{C}$ with a single pole. It can be split into a maximal chain of morphisms of compact Riemann surfaces.

$$X \overset{\beta_0}{\to} \dots \overset{\beta_n}{\to} \mathbb{P}^1\mathbb{C}$$

Maximality implies for each sub-function to have its monodromy primitive. Let's take β_i, β_{i+1} with

$i \in \{0, ..., n-1\}$. By a result of Guralnick and Thompson we have that the monodromy of $\beta_{i+1} \circ \beta_i$ has its composition factors inside the union of composition factors of the monodromy of the functions β_i and β_{i+1}.

So we just have to prove that for each i in 0, ..., n-1 , β_i has the composition factor of its monodromy being in the list.

β has a single pole, so, for each β_i there's a point P_i such the ramification degree of β_i above it equals β_i's total degree. The monodromy of β_i is a factor of the fundamental group $\pi(X_i \backslash R_i\}$ where R_i is the set of ramification points corresponding to β_i, X_i. So, it contains a cyclic permutation corresponding to a loop around P_i (that is, imagine the case of a tree, the circulation of the edges all 'around' the tree, i.e. permutations generated around ∞).

We have so reduced our work to the case of primitive permutations containing a cyclic permutation.

If G is solvable then the composition factors are cyclic.

If not, by Burnside (1911) (every non-solvable transitive group of prime degree is doubly transitive) and Schur (1933) (every primitive transitive group of composite degree which contains a cyclic permutation is doubly transitive). We can apply the following (Feit):

Let G be a non-solvable doubly transitive group of degree n which contains a cyclic permutation, then we have :

(a) $G \simeq A_n$ or S_n

or

(b) n=11 and $G \simeq PSL_2(\mathbb{F}_{11})$ or M_{11}

or

(c) n=23 and $G \simeq M_{23}$

or

(d) n=$\frac{q^m-1}{q-1}$ and $G \simeq \Psi$ where Ψ is a subgroup of $P\Gamma L_m(\mathbb{F}_q)$[4] containing $PSL_m(\mathbb{F}_q)$. When $m \geq 3$ that is also the group of collineations of $\mathbb{P}^{m-1}(\mathbb{F}_q)$ □

4.7. Bestiary

We will now see two examples.

EXAMPLE 4.7.1. We take a tree T with valency lists $V_+ = [1, 1, 1]$ and $V_- = [2, 2]$. That is the tree pictured just before. We will construct our systems $\tilde{R}$ and then R. We have

$$\tilde{R} = \begin{cases} \tilde{P}_1(z) = z + C_{1,0} \\ \tilde{P}_2(z) = z + C_{2,0} \\ \tilde{P}_3(z) = z + C_{3,0} \\ \tilde{Q}_1(z) = z^2 + D_{1,1}.z + D_{1,0} \\ \tilde{Q}_2(z) = z^2 + D_{2,1} + D_{2,0} \end{cases}$$

Choosing i_0 as 1 we get for R:

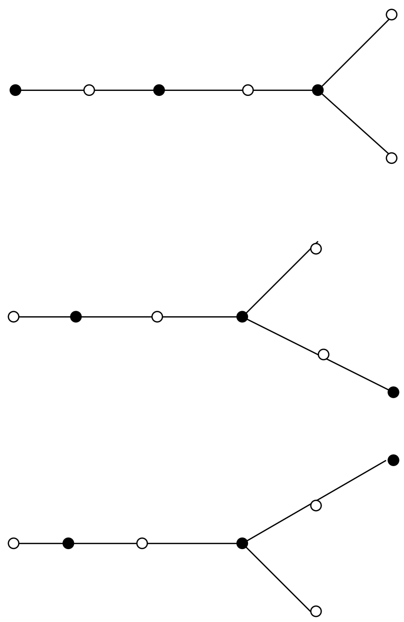
$$\begin{cases} P_1(z) = z \\ P_2(z) = z + C_{2,0} \\ P_3(z) = z + C_{3,0} \\ Q_1(z) = z^2 + D_{1,1}.z + D_{1,0} \\ Q_2(z) = z^2 + D_{2,1} + D_{2,0} \end{cases}$$

[4] $P\Gamma L_m(\mathbb{F}_q)$ is called the semilinear group; it is the extension of $PGL_m(\mathbb{F}_q)$ by the Frobenius automorphism.

We fix $C_{2,0}$ to -1 to fix automorphisms. This system has been solved, minimising the field of definition(by Schneps), using the Groebner basis algorithm. It gives three solutions corresponding to the roots of the polynomials

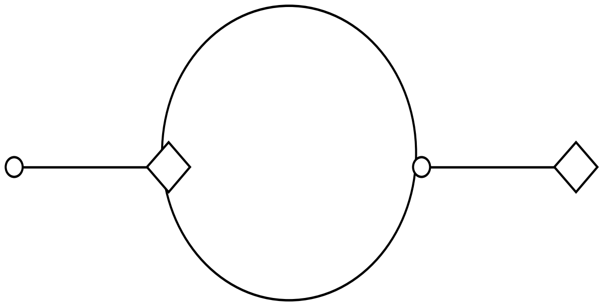
$$Q(z) = 25z^3 - 6z^2 - 6z - 2$$

To each root ρ of Q will be associated values to our indeterminates. ρ takes one real value and two complex conjugate values. They give birth to our original dessins and the two here under. (note that conjugating induces an axial symetry around the real axis.)



It is easy to see that these are the only trees of valencies [1,1,1],[2,2] and thus here valency class and galois orbit are the same. This case is 'very' common. A problem is to quantify the 'very' !

EXAMPLE 4.7.2. For this example we use a direct method.



β is a rational function, whose pre-images of 0 and 1 are opposites (1). It has a pole in 0 of order 2 (= 2.1). Let's fix the pre-image of zero at 1 and -3 of multiplicities 3 and 1. Beta has a general form :

$$\beta(x) = \frac{(x+3)(x-1)^3}{\mu.x}$$

Condition 1 must be verified :

$$\Rightarrow (x+3)(x-1)^3 - \mu.x = (x-3)(x+1)^3$$

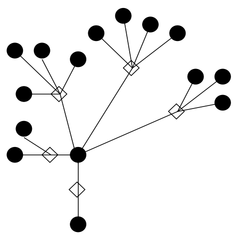
$$x^4 - 6x^2 + 8x - 3 - \mu x = x^4 - 6x^2 - 8x - 3$$

So finally:

$$\beta(x) = \frac{(x+3)(x-1)^3}{16x}$$

REMARK. We can note that the degree above ∞ is 2.(4-1)=6. (once for + and once for - ∞)

EXAMPLE 4.7.3. We come back here to one of our early dessins, I mean Leila's flowers.



We have $V_+ = [0,1,1,1,1,1]$ and $V_- = [15,0,0,0,1]$. First let us have a look at the valency class. A simple construction shows that the only trees with the same valencies are all the trees with the trees obtained by permutation of the final clusters. There are, not forgetting cyclicity invariance : $\frac{5.4.3.2.1}{5} = 24$. The same grobner basis technique gives twelve different beliy polynomials generating unequivalent trees. The # of the galois orbit we already have is thus 12. It appears that these twelve elements are the trees obtained by permutations of the 3,4,5,6-clusters by elements of A_4. And this is till now unexplained. The twelve other trees happen to be in the same orbit. So the valency orbit splits the galois orbits; that was the unusual case.

Appendix

Sketch of the dessin through Mathematica©

Having a 'small' Beliy function, it is often interesting to have an idea of how the dessins looks like. These set of commands in Mathematica© gives a sketch of the dessin given the Beliy function ϕ. The only thing mathematica does is calculating explicitly the solutions of $\phi(z) = \lambda$ for λ ranging between 0 and 1. The λ's are taken uniformly on [0,1] but the image concentrates the point round the pre-images of 1 as

$$\lim_{n \to +\infty} (x)^{1/n} = 1$$

where $x^{1/n}$ is defined over]0,1].

The image obtained is thus more likely a sketch of the $\overrightarrow{10}$[5] tangential base point as defined by Deligne (*Le groupe fondamental de la droite projective moins trois points*, In *Galois Groups over* $\mathbb{Q}$, editors : Ihara, Ribet, Serre; Math. Sci. Res. Inst. Publ. 16, Springer-Verlag)

ALGORITHM 4.7.4. We use the // sign to denote a comment

Iterations=6; // = number of λ's; as explained before there is no use in taking λ big.

Beliy[_x]=x^ 5; // = Beliy function studied, here a 5-star

K[_j]= Transpose[Function[x,{Re[x],Im[x]}][x/. NSolve[Beliy[x]==j+1,x]]]; // calculation of roots

Resul=Flatten[Table[K[(x-1)/Iterations],{x,1,Iterations+1}],1]; //nothing interesting

ListPlot[Resu, Axes->False,Prolog->Pointsize[.015]] //the plot

[5] At least it starts from one and shows the direction to follow to go to zero...

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