ON A QUESTION OF PISIER

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To the memory of Leon Ehrenpreis.

Abstract. We show that the extension by zero of a function in $M_{d\geq 3}$ has the same norm as the original function, hereby answering a question of Pisier [4].

1. Introduction

In 1950, Mahlon M. Day [1] and Jacques Dixmier [2] showed independently that for a locally compact group G, the amenability of G implies that every uniformly bounded representation is unitarisable. In loc. cit., Dixmier asked whether every group was unitarisable and, if not, whether unitarisability implies amenability. In 1955, Leon Ehrenpreis and Friedrich Mautner [3] gave a negative counterexample to the first question; they proved that $SL_2(\mathbb{R})$ is not unitarisable. The second question has so far not been entirely answered eventhough much progress has been made over the years by e.g. Bozejko, Haagerup, and Pisier - see [4].
Recently, Gilles Pisier [4] introduced the spaces of multipliers, $M_d(G)$, whose study allowed him to find a quantitative criterion under which unitarisability implies amenability. The space of multipliers $M_d(G)$ of a group G naturally generalizes Herz-Schur multipliers which correspond to the case $d = 2$; they are defined as follows:

Definition 1.1. The *space of multipliers*, $M_d(G)$, consists in all functions $f : G \to \mathbb{C} = B(\mathbb{C}, \mathbb{C})$ which can be factored by bounded functions $\xi_i : G \to B(\mathcal{H}_i, \mathcal{H}_{i-1})$ as

$$f(g) = f(g_1 \dots g_d) = \xi_1(g_1) \dots \xi_d(g_d)$$

where the $\mathcal{H}_i$ are Hilbert spaces and $\mathcal{H}_0 = \mathcal{H}_d = \mathbb{C}$, and the ξ_i do not depend on the decomposition of g.

1.1. Extensions by 0.

Let G be a locally compact group and H a closed subgroup. For a continuous function $f : H \to \mathbb{C}$, we define $\tilde{f} : G \to \mathbb{C}$ as the extension of f identically 0 outside of H. Pisier [4] notes that the norm of f and $\tilde{f}$ are the same for $d = 2$ and ask whether this holds in general:

Question 1.2. [4, Section 2] *Is it true that*

$$\|f\|_{M_d(H)} = \|\tilde{f}\|_{M_d(G)}$$

for $d > 2$?

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2. Solution

Factorizion of the extension. We will first construct of factorization of $\tilde{f}$ which will prove useful in the rest of the paper. Let $f \in M_d(H)$, where $d \geq 2$ for $H \subset G$. We will extend f by 0 :

$$\tilde{f} : G \to \mathbb{C} : g \mapsto \begin{cases} f(g) & g \in H \\ 0 & g \notin H \end{cases}$$

Assume that f factors through $\xi_{1,\ldots,d}$:

$$f(t_1 \ldots t_d) = \xi_1(t_1) \ldots \xi_d(t_d)$$

where $\xi_d : \mathcal{H}_{d-1} \to \mathcal{H}_d$, et $\mathcal{H}_0 = \mathcal{H}_d = \mathbb{C}$. We define the auxilliary functions $\tilde{\xi}_i$:

$$\tilde{\xi}_i(t) = \begin{cases} \xi_i(t) & t \in H \\ 0 & t \notin H \end{cases}$$

We will later define operators $\Xi_i = \operatorname{Ind}_H^G \tilde{\xi}_i$ whose construction is analog to that of the induced representation.

2.1. **Main result.**

Proposition 2.1. *The functions f and $\tilde{f}$ have the same norm:*

$$\|f\|_{M_d(H)} = \|\tilde{f}\|_{M_d(G)}$$

Proof. It is clear that $\|f\|_{M_d(H)} \leq \|\tilde{f}\|_{M_d(G)}$ it is therefor sufficient to show the opposite inequality.

Fix a set of right H-coset representatives in G, i.e. $G = H \sqcup Hg_1 \ldots \sqcup Hg_k$, and call it $\mathcal{C} = \{e, g_1, \ldots, g_k\}$.
Consider now the following operator which are given in block-form :

$$\begin{aligned} \Xi_1(t) &= \left(\tilde{\xi}_1(tg^{-1})\right)_{g \in \mathcal{C}} \\ \Xi_{\lambda=2,\ldots,d-1}(t) &= \left(\tilde{\xi}_\lambda(gt\tilde{g}^{-1})\right)_{g,\tilde{g} \in \mathcal{C}} \\ \Xi_d(t) &= \left(\tilde{\xi}_1(gt)\right)^t_{g \in \mathcal{C}} \end{aligned}$$

The domains and ranges are $\Xi_1 : \mathbb{C} \to \mathcal{H}_1^{[G:H]}$, $\Xi_{\lambda=2,\ldots,d-1} : \mathcal{H}_{\lambda-1}^{[G:H]} \to \mathcal{H}_\lambda^{[G:H]}$ and $\Xi_d : \mathcal{H}_{d-1}^{[G:H]} \to \mathbb{C}$. These Ξ factor $\tilde{f}$ (see lemma 2.2) :

$$\tilde{f}(t_1 \ldots t_d) = \Xi_1(t_1) \ldots \Xi_d(t_d)$$

We will show that $\|\Xi_k\| = \|\xi_k\|$. The cases $k = 1, d$ are trivial we will thus focus on the central Ξ's.
Let g be an element of G, for a given $g_i \in \mathcal{C}$ there exists a unique $g_j \in \mathcal{C}$ such that $g \in g_i H g_j^{-1}$. Therefore, Ξ_i is a permutation matrix and its norm is the supremum of the norm of its entries. Since these entries are in ξ_i, the norms of the lower case and upper case operators are the same. □

Lemma 2.2. *Let the function $\tilde{f}$ and the operators Ξ_d be as above. We have the following Schur factorization :*

$$\tilde{f}(t_1 \ldots t_d) = \Xi_1(t_1) \ldots \Xi_d(t_d)$$

Proof. The element t_1 belongs to a unique $Hg_{(1)}$. Given this element $g_{(1)} \in \mathcal{C}$, t_2 belongs to a unique $g_{(1)}^{-1} H g_{(2)}$. Inductively, all $t_{i=2\dots d-1}$ belong to a unique $g_{(i-1)}^{-1} H g_{(i)}$. Therefore,

$$\begin{aligned}
& \Xi_1(t_1) \cdots \Xi_2(t_2) \dots \Xi_{d-1}(t_{d-1}) \\
= \quad & \left(\tilde{\xi}_1(t_1 g^{-1})\right)_{g \in \mathcal{C}} \cdot \left(\tilde{\xi}_2(g t_2 h^{-1})\right)_{g,h \in \mathcal{C}} \cdot \Xi_2(t_3) \dots \Xi_{d-1}(t_{d-1}) \\
= \quad & \left(\sum_{g \in \mathcal{C}} \tilde{\xi}_1(t_1 g^{-1}) \cdot \tilde{\xi}_2(g t_2 h^{-1})\right)_{h \in \mathcal{C}} \cdot \Xi_2(t_3) \dots \Xi_{d-1}(t_{d-1}) \\
= \quad & \left(\xi_1(t_1 g_{(1)}^{-1}) \cdot \tilde{\xi}_2(g_{(1)} t_2 h^{-1})\right)_{h \in \mathcal{C}} \cdot \Xi_2(t_3) \dots \Xi_{d-1}(t_{d-1}) \\
\vdots \quad & \\
= \quad & \left(\xi_1(t_1 g_{(1)}^{-1}) \cdot \xi_2(g_{(1)} t_2 g_{(2)}^{-1}) \dots \tilde{\xi}_{d-1}(g_{(d-1)} t_d h^{-1})\right)_{h \in \mathcal{C}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \Xi_1(t_1) \cdots \Xi_2(t_2) \dots \Xi_d(t_d) \\
= \quad & \left(\xi_1(t_1 g_{(1)}^{-1}) \cdot \xi_2(g_{(1)} t_2 g_{(2)}^{-1}) \dots \tilde{\xi}_{d-1}(g_{(d-1)} t_d h^{-1})\right)_{h \in \mathcal{C}} \left(\tilde{\xi}_d(h t_d)\right)^t_{h \in \mathcal{C}} \\
= \quad & \sum_{h \in \mathcal{C}} \xi_1(t_1 g_{(1)}^{-1}) \dots \tilde{\xi}_{d-1}(g_{(d-1)} t_d h^{-1}) \tilde{\xi}_d(h t_d) \\
= \quad & \xi_1(t_1 g_{(1)}^{-1}) \dots \tilde{\xi}_{d-1}(g_{(d-1)} t_d g_{(d)}^{-1}) \tilde{\xi}_d(h t_d) \\
= \quad & \begin{cases} \xi_1(t_1 g_{(1)}^{-1}) \dots \xi_{d-1}(g_{(d-1)} t_d g_{(d)}^{-1}) \xi_d(g_{(d)} t_d) & \text{if} \quad t_d \in g_{(d)}^{-1} H \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The second case occurs exactly when $t_1 \dots t_d \notin H$. The fist case corresponds to $t_1 \dots t_d \in H$. Now, we note that the last line is nothing but

$$\begin{aligned}
& \xi_1(t_1 g_{(1)}^{-1}) \dots \xi_{d-1}(g_{(d-1)} t_d g_{(d)}^{-1}) \xi_d(g_{(d)} t_d) \\
= \quad & f(t_1 g_{(1)}^{-1} \cdot \dots xi_{d-1}(g_{(d-1)} t_d g_{(d)}^{-1}) \xi_d(g_{(d)} t_d) \\
= \quad & f(t_1 t_2 \dots t_d)
\end{aligned}$$

. □

Example in M_3 :
Take the extension $H = \{0\} \subset \mathbb{Z}/2\mathbb{Z}$ and assume $f(0) = 3 \in \mathbb{C}$.
The function f factors through

$$\begin{aligned}
\xi_1(0) : \quad & \mathbb{C} \to \mathcal{H}_1 = \mathbb{C} & : z \mapsto 3.z \\
\xi_2(0) : \quad & \mathcal{H}_1 = \mathbb{C} \to \mathcal{H}_2 = \mathbb{C} & : z \mapsto z \\
\xi_3(0) : \quad & \mathcal{H}_2 = \mathbb{C} \to \mathbb{C} & : z \mapsto z.
\end{aligned}$$

The function $\tilde{f}$ is simply defined by $\tilde{f}(0) = 3$ and $\tilde{f}(1) = 0$. As $\mathbb{Z}/2\mathbb{Z} = 1 + H \sqcup 0 + H$, the set $\mathcal{C}$ contains two elements : 0 and 1. Therefore:

- $\Xi_1(t) := \begin{pmatrix} \tilde{\xi}_1(t-0) & \tilde{\xi}_1(t-1) \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_1(t) & \tilde{\xi}_1(t+1) \end{pmatrix}$

- $\Xi_2(t) := \begin{pmatrix} \tilde{\xi}_2(0+t-0) & \tilde{\xi}_2(0+t-1) \\ \tilde{\xi}_2(1+t-0) & \tilde{\xi}_2(1+t-1) \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_2(t) & \tilde{\xi}_2(t+1) \\ \tilde{\xi}_2(t+1) & \tilde{\xi}_2(t) \end{pmatrix}$
- $\Xi_3(t) := \begin{pmatrix} \tilde{\xi}_3(0+t) & \tilde{\xi}_3(1+t) \end{pmatrix}^t = \begin{pmatrix} \tilde{\xi}_3(t) & \tilde{\xi}_3(t+1) \end{pmatrix}^t$

for the decomposition $\tilde{f}(0) = \tilde{f}(1+0+1) = f(0)$, we have

$$\begin{aligned} & \Xi_1(1) \cdot \Xi_2(0) \cdot \Xi_3(1) \\ = & \begin{pmatrix} \tilde{\xi}_1(1) & \tilde{\xi}_1(1+1) \end{pmatrix} \begin{pmatrix} \tilde{\xi}_2(0+0+0) & \tilde{\xi}_2(0+0+1) \\ \tilde{\xi}_2(0+0+1) & \tilde{\xi}_2(1+0+1) \end{pmatrix} \begin{pmatrix} \tilde{\xi}_3(1) \\ \tilde{\xi}_3(1+1) \end{pmatrix} \\ = & \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ = & \ 3 \end{aligned}$$

As expected !

References

[1] Day, M., *Means for the bounded functions and ergodicity of the bounded representations of semi-groups*, Trans. Amer. Math. Soc. 69 (1950) 276-291

[2] Dixmier, J., *Les moyennes invariantes dans les semi-groupes et leurs applications*, Acta Sci. Math. Szeged 12 (1950), 213-227

[3] Ehrenpreis, L., and Mautner, F.I., *Uniformly bounded representations of groups*, Proc. nat. Acad. Sc. U.S.A. 41 (1955) 231-233

[4] Pisier, Gilles. Are unitarizable groups amenable? Infinite groups: geometric, combinatorial and dynamical aspects, 323–362, Progr. Math., 248, Birkhäuser, Basel, 2005.

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