# ON ALTERNATE DUAL FRAMES

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Abstract. The set of alternate duals of a given frame $X = \{x_j\}_{j\in\mathbb{J}}$ in a Hilbert space $H$ carries a natural structure of affine space whose base point is the *canonical dual* of $X$. We describe this structure in terms of bounded linear idempotents of $B(l^2(\mathbb{J}))$.

## 1. Introduction

Let us fix throughout the whole paper a separable Hilbert space $H$ and a countable set $\mathbb{J}$. Now let $X = \{x_j\}_{j\in\mathbb{J}}$ be a family of vectors in $H$ indexed by $\mathbb{J}$. By definition $X$ is called a *frame* if there exist two positive constants $A, B$ such that

$$A\|x\|^2 \leq \sum_{j\in\mathbb{J}} |\langle x, x_j\rangle|^2 \leq B\|x\|^2$$

for all $x$ in $H$. Besides, the family $X$ allows to define a linear operator

$$\theta_X : x \longmapsto (\langle x, x_j\rangle)_{j\in\mathbb{J}}$$

from $H$ into the set of scalar sequences indexed by $\mathbb{J}$.

We observe that $X$ is a frame if and only if the so-called *analysis operator* $\theta_X$ is a bounded linear operator from $X$ onto a closed subspace of $l^2(\mathbb{J})$ and if the adjoint

$$\theta_X^* : l^2(\mathbb{J}) \longrightarrow H$$

is surjective. Moreover, if we denote $\{e_j\}_{j\in\mathbb{J}}$ the canonical orthonormal basis of $l^2(\mathbb{J})$, we have

$$x_j = \theta_X^* e_j$$

for all $j$ in $\mathbb{J}$.

Thus there is a natural bijection between the set of frames in $H$ indexed by $\mathbb{J}$ and the set of bounded linear operators from $H$ onto a closed subspace of $l^2(\mathbb{J})$ whose adjoints are surjective. In the remainder of this paper, the letters $X$ and $Y$ shall always stand for such frames.

In addition to this correspondence, it proves convenient to introduce the projection

$$p_X : l^2(\mathbb{J}) \longrightarrow \text{Im } \theta_X \subset l^2(\mathbb{J})$$

onto the range of $\theta_X$ along Ker $\theta_X^*$. It is worth noticing that these two operators are related by the formula

$$p_X = \theta_X(\theta_X^*\theta_X)^{-1}\theta_X^*$$

where the invertibility of $\theta_X^*\theta_X$, the *frame operator*, is guaranteed by the fact that $X$ is a frame. As observed in [2, p. 11], a frame $X$ is a *Riesz basis* if and only if its *synthesis operator* $\theta_X^*$ is injective or, equivalently, if $p_X = 1$.

We now come to the objective of this note, which is to investigate the notion of duality between frames. Since we work with their associated operators rather than with the frames themselves, we will take as a definition the characterization exhibited in [2, Proposition 1.17]:

**Definition 1.1.** A frame $Y$ is called an *alternate dual* of the frame $X$ if $\theta_Y^*\theta_X = 1$.

First note that this defines an equivalence relation on the set of frames. Then take a frame $X$ and observe that the operator $\theta := \theta_X(\theta_X^*\theta_X)^{-1}$ is bounded with closed range Im $\theta$ = Im $\theta_X$ and with surjective adjoint $\theta^* = (\theta_X^*\theta_X)^{-1}\theta_X^*$. By the correspondence above, there exists a unique frame $X'$ which admits $\theta$ as its analysis operator. It turns out that $X'$ is nothing but the *canonical dual* of $X$ (see [2] Section 1.2), which is therefore characterized by the following identity:

$$\theta_{X'} = \theta_X(\theta_X^*\theta_X)^{-1}.$$

It can be shown that the set of alternate duals of $X$ corresponds to the following affine subspace of $B(H, l^2(\mathbb{J}))$, the set of bounded linear operators from $H$ to $l^2(\mathbb{J})$:

$$\theta_{X'} + \text{Ker } \{\phi \longmapsto \phi^*\theta_X\}.$$

But the purpose of this study is to establish a correspondence between alternate duals of $X$ and certain idempotents in the algebra $B(l^2(\mathbb{J}))$ of bounded linear operators on $l^2(\mathbb{J})$. We recall that an element $p$ in $B(l^2(\mathbb{J}))$ is called an idempotent if $p^2 = p$ and that it is called a projection if $p^2 = p = p^*$.

The key point in this note is the fact that $Y$ is an alternate dual of $X$ if and only if $\theta_X\theta_Y^*$ is an idempotent (Proposition 3.2).

But this statement can be made more precise. Namely, the latter correspondence provides us with a moduli space for the set of alternate duals of $X$.

Let us first introduce a notation from [1].

**Definition 1.2.** For any idempotent $p$ in $B(l^2(\mathbb{J}))$, we denote

$$\mathcal{F}_p := p + pB(l^2(\mathbb{J}))(1-p)$$

the affine subspace of $B(l^2(\mathbb{J}))$ made of all idempotents in $B(l^2(\mathbb{J}))$ whose range is equal to the range of $p$.

We let the reader check the assertion contained in this definition (see [1, Lemma 4.1] if needed).

The main result of this note is that the mapping $Y \longmapsto \theta_X \theta_Y^*$ is a bijection from the set of alternate duals of $X$ onto $\mathcal{F}_{p_X}$ (Theorem 3.3).

Note that $p_X$ is the only projection in the latter and that it corresponds to the canonical dual $X'$ of $X$. It is well-known that $X'$is the only alternate dual of $X$ if and only if $X$ is a Riesz basis (cf. [2, Corollary 2.26]). We note in Corollary 3.4 that this is a straightforward consequence of the latter description.

In Section 2 we establish two preliminary lemmas which are meant to isolate as much as possible of what is independent from frame theory here. Then Section 3 is devoted to the proof of the results mentioned above. And finally in Section 4, we use this viewpoint to give a quantitative approach of the following questions: when $Y$ is not an alternate dual of $X$, how far is it from being so?

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## 2. Preliminaries

Here we prove two lemmas which are fairly general. They could obviously be stated for operators between, say, Banach spaces. But for the sake of this paper, we only need to work with $S, T$ bounded linear operators from $H$ to $l^2(\mathbb{J})$.

**Lemma 2.1.** *If $S^*T = 1$ then both $S$ and $T$ have closed range and a surjective adjoint.*

*Proof.* First observe that $S^*T = 1$ implies trivially the surjectivity of $S^*$. Next assume $y$ lies in the closure of Im $T$ and take $Tx_n \to y$. Then applying $S^*$, we find that $x_n = S^*Tx_n \to S^*y$. Hence $Tx_n \to TS^*y$ so $y = TS^*y$ belongs to Im $T$. This proves that $T$ has closed range. Now observe that $T^*S = (S^*T)^* = 1$, so that the preceding conclusions also apply to $T$ and $S$ in the reverse order. □

**Lemma 2.2.** *The identity $S^*T = 1$ holds if and only if the following three conditions hold:*

*(i) $S^*$ is surjective;*
*(ii) $T$ is injective;*
*(iii) $TS^*$ is idempotent.*

*Proof.* Assume $S^*T = 1$ first. As observed in the proof above, it follows readily that both $S^*$ and $T^*$ are surjective. Now recall Ker $T = (\text{Im } T^*)^\perp = \{0\}$, so that in particular the surjectivity of $T^*$ implies (ii). Property (iii) follows from a straightforward computation.
Conversely, assume that the three conditions are fulfilled. Then set $p := S^*T$ and $q := TS^*$. It follows from (i) that Im $q =$ Im $T$. So in particular by (iii), $q$ is equal to the identity on Im $T$, i.e. $T = qT$. On the other hand, we have $qT = TS^*T = Tp$ so that $T(1-p) = 0$. We conclude from (ii) that $1 - p = 0$, i.e. $S^*T = 1$. □

## 3. Alternate duals and idempotents

We will now draw some consequences which are more specific to frames.

First we observe that the duality condition with respect to a frame $X$ can be satisfied by nothing but operators which automatically fall in the class of analysis operators for frames.

**Proposition 3.1.** *Let $X$ be a frame and let $S : H \longrightarrow l^2(\mathbb{J}))$ be a bounded linear operator. It $S^*\theta_X = 1$, then there exists a unique frame $Y$ such that $S = \theta_Y$. Moreover, $Y$ is an alternate dual of $X$.*

*Proof.* This follows essentially from Lemma 2.1, *via* the correspondence exhibited in the introduction. □

Here is now the key observation.

**Proposition 3.2.** *Let $X$ and $Y$ be two frames. Then $Y$ is an alternate dual of $X$ if and only if $\theta_X\theta_Y^*$ is an idempotent of $B(l^2(\mathbb{J}))$.*

*Proof.* Since $X$ are $Y$ are frames, we know that $\theta_X^*$ and $\theta_Y^*$ are surjective. And since Ker $\theta_X = (\text{Im } \theta_X^*)^\perp$, we also have in particular that $\theta_X$ is injective. Thus the result is nothing but Lemma 2.2 applied to $\theta_Y$ and $\theta_X$. □

More precisely, we have:

**Theorem 3.3.** *The mapping $Y \longmapsto \theta_X\theta_Y^*$ is a bijection from the set of alternate duals of $X$ onto $\mathcal{F}_{p_X} = p_X + p_X B(l^2(\mathbb{J}))(1 - p_X)$.*

*Proof.* Let $Y$ be an alternate dual of $X$, so that $q := \theta_X\theta_Y^*$ is an idempotent by straightforward computation. Also Im $q =$ Im $\theta_X$ by surjectivity of $\theta_Y^*$. Since Im $\theta_X =$ Im $p_X$, it follows from Definition 1.2 that $q$ belongs to $\mathcal{F}_{p_X}$. Thus the mapping under consideration does take its values $\mathcal{F}_{p_X}$.
Now let $q$ be an idempotent in $\mathcal{F}_{p_X}$ and set $S := q^*\theta_X(\theta_X^*\theta_X)^{-1}$. This defines a bounded operator $S : H \longrightarrow l^2(\mathbb{J})$ which satisfies $\theta_X S^* = p_X q$ and $S^*\theta_X = (\theta_X^*\theta_X)^{-1}\theta_x^* q\theta_X$ by definition of $S$ and $p_X$. Since Im $q =$ Im $p_X =$ Im $\theta_X$, we have $p_X q = q$ and $q\theta_X = \theta_X$. Hence $\theta_X S^* = q$ and $S^*\theta_X = 1$. By Proposition 3.1, there exists a unique frame $Y$ such that $\theta_Y = S$. Then $Y$ is an alternate dual of $X$ which is mapped onto $\theta_X\theta_Y^* = q$. The proof is complete.

□

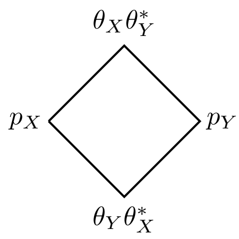
**Corollary 3.4.** *The canonical dual $X'$ is the only dual of $X$ if and only if $X$ is a Riesz basis.*

*Proof.* Since $X$ is a frame, it appears that $p_X$ is necessarily nonzero and therefore the subspace $p_X B(l^2(\mathbb{J}))(1-p_X)$ is equal to $\{0\}$ if and only if $p_X = 1$. So the first assertion is equivalent to $p_X = 1$ by Theorem 3.3. As observed in the introduction, the latter condition is equivalent to the second assertion. □

## 4. Application

We recall from [1] that given two idempotents $p, q$, we denote $k(p, q)$, if it exists, the idempotent $k$ which is determined by Im $k$ = Im $p$ and Ker $k$ = Ker $q$.

Assume that $Y$ is an alternate dual of $X$. Then we have four idempotents $p_X$, $p_Y$, $\theta_X\theta_Y^*$ and $\theta_Y\theta_X^*$ whose ranges and nullspaces are intertwined in a way which can be recorded in the following diagram, according to the conventions introduced in [1]:




This reads as follows: $p_X$ and $\theta_X\theta_Y^*$ (respectively $p_X$ and $\theta_Y\theta_X^*$) share the same range (respectively nullspace), like $\theta_Y\theta_X^*$ and $p_Y$ (respectively $\theta_X\theta_Y^*$ and $p_Y$) do.

In other terms $k(p_X, p_Y)$ and $k(p_Y, p_X)$ exist, and that they are given in this case by the formulas $k(p_X, p_Y) = \theta_X\theta_Y^*$ and $k(p_Y, p_X) = \theta_Y\theta_X^*$.

But $k(p_X, p_Y)$ and $k(p_Y, p_X)$ exist in general, even when $Y$ is not an alternate dual of $X$. This relies essentially on the fact that $p_X$ and $p_Y$ are two projections, and not only idempotents. In particular, the so-called Kovarik formula $k(p_X, p_Y) = p_X(p_X + p_Y - 1)^{-2}p_Y$ holds for every frames $X$, $Y$. We refer to [1] for a detailed proof of these statements.

We deduce from this discussion that Proposition 3.2 can be restated as follows:

**Proposition 4.1.** *Let $X$, $Y$ be two frames. Then $Y$ is an alternate dual for $X$ if and only if $\theta_X\theta_Y^* = p_X(p_X + p_Y - 1)^{-2}p_Y$.*

As a conclusion, we point out that the quantity

$$\|\theta_X\theta_Y^* - p_X(p_X + p_Y - 1)^{-2}p_Y\|$$

could measure how far $Y$ is from being an alternate dual of $X$.

## References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